East model: Aging through Hierarchical Coalescence

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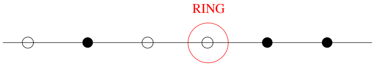
Overview of the talk

1. I Part : the East model
   - East model : definition and main features.
   - Previous results on the relaxation to the equilibrium.
   - High density, non-equilibrium dynamics.
   - Staircase behavior of the density ;
   - Aging for density-density time auto-correlation ;
   - Scaling limit for the inter-vacancy interval length.
2. II Part : Hierarchical Coalescence Process.
   - Main features of a hierarchical coalescence process.
   - Universality for the HCP.
   - Implications for the East model :
3. Open problems.

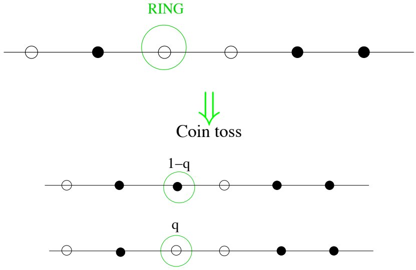
East Model [Jackle-Eisinger '91]

- Configuration space $\Omega = \{0,1\}^{\mathbb{Z}}$
- $\sigma \in \Omega, \quad \sigma(x) \in \{0,1\}$
  - 1 : there is a particle at site x
  - 0 : there is no particle at site x

- Glauber dynamics with kinetic constraint : the right neighbor should be empty for a flip to take place.



Nothing happens



Main Features of the East Model

- $\sigma \in \Omega := \{0,1\}^{\mathbb{Z}}, \quad \forall x \in \mathbb{Z} : \sigma(x) \in \{0,1\}$
- $\{\sigma_t\}_{t\geq 0}$ denotes the process with initial config. σ
- $$\mathcal{L}f(\sigma) := \sum_{x\in\mathbb{Z}} (1-\sigma(x+1))(\mu_x(f) - f(\sigma))$$
- Reversible w.r.t. μ (Bernoulli product measure of density $p = 1-q$).
- Introduced by physicists to model liquid/glass transition.
- Constraint on the allowed moves (more effective as $q \downarrow 0$) simulates *geometric constraints* on molecules in highly dense liquids.
- As $q \downarrow 0$ constraints slow down the dynamics. The liquid freezes into an amorphous solid state $\sim$ glass.

Relaxation to equilibrium : basic results

Definition

$$T_{\rm rel} := (\text{spectral gap of } \mathcal{L})^{-1}$$

Theorem

- $T_{\rm rel} < +\infty \quad \forall q \in (0,1)$ *(Aldous, Diaconis '02)*
- $T_{\rm rel} \simeq \left(\frac{1}{q}\right)^{|\log_2(q)|/2}$ *as* $q \downarrow 0$ *(Cancrini, M., Roberto, Toninelli '08)*

Theorem (Cancrini, M., Schonmann, Toninelli. '09)

Let $\nu \neq \mu$ *be a different product measure. Then*

$$|E_\nu(f(\sigma_t)) - \mu(f)| \le C_f \exp(-t/2T_{\rm rel})$$

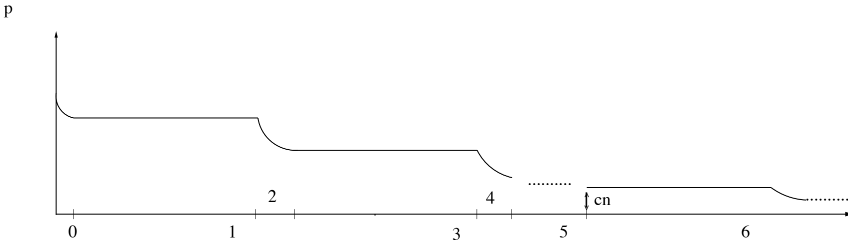
Basic Setting

1. Start at time zero from a quite general *renewal law* ν independent from q ;
2. Run the Glauber dynamics with $q \ll 1$;
3. Focus on the *pre-asymptotic* behavior not too far from the origin (or in large q-independent interval) and up to time scales $T = T(q)$ with $1 \ll T \ll T_{\mathrm{rel}}$.
4. Physically we are considering a *quench* from *low* to *high* density, i.e. from the liquid to the glass phase ;

Plateau behavior of the density

- Numerical simulations and non-rigorous theoretical analysis suggest that, as $q \to 0$:

  (a) the model does not reach equilibrium ;
  (b) time auto-correlation function shows aging ;
  (c) the density profile exhibits plateau behavior.



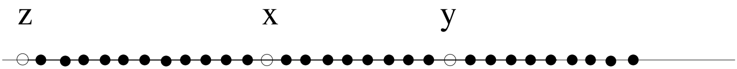
A First Key Observation

As $q \downarrow 0$:

1. Dynamics is dominated by killing the excess vacancies.
2. Dynamics $\Longleftrightarrow$ coarsening of intervals delimited by consecutive vacancies.
3. To kill a vacancy at x the dynamics *must* bring a vacancy at $x+1$ from the nearest (East-ward w.r.t. x) vacancy $\Rightarrow$ *cooperative* relaxation.
4. Cooperative relaxation requires a certain number of auxiliary *extra* vacancies to be first created and then destroyed $\Rightarrow$ *Energy Barriers*.
5. *Metastable* effects very relevant.
6. Key question : what is the *structure* of the energy barriers ?

Energy Barriers Structure and Activation Times

$$y - x \in [2^{n-1}+1, 2^n], \qquad 1 \leq n \ll \log_2(1/q)$$



Combinatorial argument (Evans, Sollich + Chung-Diaconis-Graham 01) : during the killing of vacancy at x *at least* n extra vacancies between x and y.

Energy barrier $\Delta E_n = n \Rightarrow$ *Activation Time* $t_n := (1/q)^n$.

Metastability : Actual killing is *random and istantaneous* (w.r.t. to the expected time t_n) and occurs on scale $t_{n-1} = \frac{1}{q^{n-1}}$.

Killing vacancy at $x \;\Leftrightarrow$ coalescing domain $[x, y]$ with $[z, x]$

Active and stalling periods

Definition (Hierarchy of activation times)

$$t_n^+ := t_n^{1+\epsilon} = \left(\frac{1}{q^n}\right)^{1+\epsilon}, \qquad t_n^- := t_n^{1-\epsilon} = \left(\frac{1}{q^n}\right)^{1-\epsilon}$$

Definition

Domain $[x, x+\ell]$ is of *class* n if $\ell \in [2^{n-1}+1, 2^n]$, $n \geq 1$.
Vacancy at x is of class n if it is the left border of a domain of class n.

Definition

- $[t_n^-, t_n^+]$ is called the n-th *active period* ;
- $[t_n^+, t_{n+1}^-]$ is called the n-th *stalling* period (nothing happens).

Evolution during the n-th active period.

- Recursively assume that at time t_n^- all vacancies in e.g. $[-2^N, 2^N]$, $N \gg 1$, are of class at least n w.h.p.
- Vacancies of class *larger* than n do not disappear w.h.p.
- When a vacancy of the n-th class disappears $\Leftrightarrow$ the class of the vacancy to its *left* becomes at least $n+1$ $(2^n + 2^n = 2^{n+1})$.

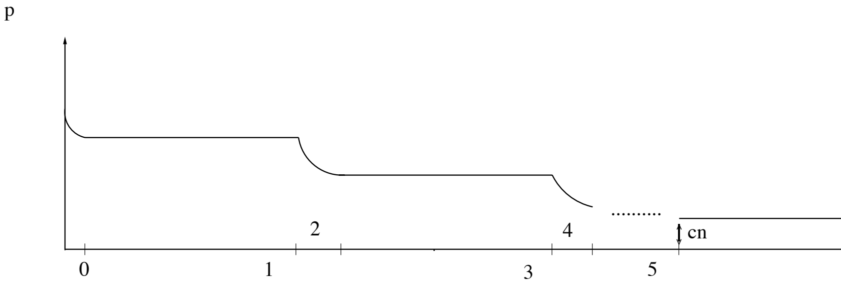
- Thus vacancies of class n either disappear *directly* or *increase* their class w.h.p.
- At the end of the period (t_n^+) w.h.p. *all* vacancies are of class at least $n+1$ and were already present at t_n^- ;
- Between the *stalling* period $[t_n^+, t_{n+1}^-]$ w.h.p. no vacancy is killed $\Rightarrow$ the recursion step is proved.

Staircase Behavior

Theorem

$\forall \sigma \in \{0,1\}^{\mathbb{Z}}$, $\forall n \in \mathbb{N}$ there exists $c_n(\sigma)$ s.t.

$$\lim_{q \downarrow 0} \sup_{t \in [t_n^+, t_{n+1}^-]} |\mathbb{P}_\sigma(\sigma_t(0) = 0) - c_n(\sigma)| = 0$$



Heigth of plateaux

Definition

Let ν be a (non-trivial) probability law over the integers with finite mean. We define Ren(ν) as the stationary renewal distribution on $\mathbf{Z}$ with interval law ν

Theorem

If the initial distribution is $Q = Ren(\nu)$ *then*

$$\lim_{n\to\infty} \lim_{q\downarrow 0} \mathbb{E}_Q(c_n)(2^n + 1) = 1$$

Coarsening

Let $\tilde{X}$ be the non-negative random variable such that

$$\mathbb{E}(\exp(-s\tilde{X})) = 1 - \exp\left\{ - \int_1^\infty \frac{e^{-sx}}{x} dx \right\} \qquad s > 0$$

Theorem

If the initial distribution is $Q = Ren(\nu)$ then for any bounded function f and any $k \in \mathbb{Z}$

$$\lim_{n\uparrow\infty} \lim_{q\downarrow 0} \sup_{t\in[t_n^+, t_{n+1}^-]} \left| \mathbb{E}_Q\left(f(X_k^{(n+1)}(t))\right) - \mathbb{E}\left(f(\tilde{X})\right) \right| = 0$$

where $X_k^{(n)}(t) := (x_{k+1}(t) - x_k(t))/(2^{n-1} + 1)$

$\Longrightarrow$ Mean domain length grows as $t^{\log 2/|\log q|}$

Aging

Theorem

Fix $\sigma \in \{0,1\}^{\mathbb{Z}}$, $\forall n, m \in \mathbb{N}$ *there exists* $c_{n,m,x}(\sigma)$ *s.t.*

$$\lim_{q \downarrow 0} \sup_{t \in [t_n^+, t_{n+1}^-]} \sup_{s \in [t_m^+, t_{m+1}^-]} \left| \mathrm{Cov}_\sigma(\sigma_t(x); \sigma_s(x)) - c_{n,m,x}(\sigma) \right| = 0$$

If $Q = Ren(\nu)$ *set* $\rho_x := Q(\sigma(x) = 0)$, *then*

$$\lim_{n \to \infty} \lim_{m \to \infty} \lim_{q \downarrow 0} \left| \mathbb{E}_Q(c_{n,m,x}) - \frac{\rho_x}{(2^n+1)} \left(1 - \frac{\rho_x}{(2^m+1)} \right) \right| = 0$$

Hierarchical Coalescence Process (HCP)

- Main motivation : physical modeling of non-equilibrium coarsening dynamics of 1D systems (Derrida, Bray, Godreche,...) ;
- Features : infinite sequence of 1D coalescence processes $\{\xi_n\}_{n=1}^{\infty}$ with "end of ξ_n = beginning of ξ_{n+1}".
- The n-th process live in the n-th *epoch* and *only* intervals with length $\in [d_n, d_{n+1})$ are *active* i.e. merge with very general rates with the left or right neighbor.
- We assume $d_n \uparrow$ and $2d_n \geq d_{n+1}$.
- Examples : $d_n = n$ ("Paste-all-model"), $d_n = a^n,\ a \in (1, 2]$, $d_n = 2^{n-1} + 1 \Leftrightarrow$ the East model.

Some Facts

1. Assume that at the beginning of the first epoch the intervals form a renewal process
2. Physicists observed a large scale (i.e. $n \to \infty$) universality of rescaled variables (e.g. (domain length)$/d_n$).
3. Different models show the same behavior independently of the merging rates and of the active domains range. For example the "Paste-all-model" behaves as the East model a fact refereed to as "very surprising" in the physics literature.
4. Physicists were not able to prove the scaling hypothesis nor were able to classify the universality classes according to the initial renewal process.

HCP definition I

$\Omega^{(n)}$ is the subset of $\Omega := (0,1)^{\mathbb{Z}}$ with each domain (=interval among consecutive zeros) of class at least n.
Class 0 if $d = 1$, class $n \geq 1$ if $d \in [d_n, d_{n+1})$.

n-th epoch Coalescence Process

Initial configuration : $\sigma_0^{(n)} \in \Omega^{(n)}$
Dynamics :

- independent exponential clock on each domain $[x, y]$ with rate $\lambda_n(y - x)$;
- $\lambda_n(d) > 0$ iff $d \in [d_n, d_{n+1})$;
- when clock rings on $[x, y]$ and if the domain is still present $\Rightarrow$ incorporate left domain, i.e. erase point x.

At infinite time the configuration $\sigma_\infty^{(n)}$ belongs to $\Omega^{(n+1)}$

HCP definition II (infinite epochs)

1. Start from $\sigma \in \Omega = \Omega^{(0)}$ and run the 0-th epoch coalescence process for an infinite time to get $\sigma_\infty^{(0)}$;
2. Start from $\sigma_\infty^{(0)} \in \Omega_1$ and run the 1-th epoch coalescence process for an infinite time to get $\sigma_\infty^{(1)}$;
3. ...
4. Start from $\sigma_\infty^{(n-1)} \in \Omega^{(n)}$ and run the n-th epoch coalescence process for an infinite time to get $\sigma_\infty^{(n)}$
5. ...

HCP Main Results

Theorem

Assume $\sigma \sim Q = Ren(\nu)$ with ν any finite mean probability measure on $[1, \infty)$. Then (i) $\sigma_t^{(n)}$ is distributed with $Ren(\nu_t^{(n)})$; (ii) Let $X^{(n)}$ be distributed with $\nu_0^{(n)}$ and $Z^{(n)} := X^{(n)}/d_n$. Then $Z^{(n)}$ weakly converges to $Z^{(\infty)}$ with

$$\mathbb{E}(e^{-s\,Z^{(\infty)}}) = 1 - \exp\left\{ - \int_1^\infty \frac{e^{-sx}}{x} dx \right\}$$

(iii) If instead $Q = Ren(\nu|0)$ with ν in the domain of attraction of an α-stable law then

$$\mathbb{E}(e^{-s\,Z^{(\infty)}}) = 1 - \exp\left\{ - \alpha \int_1^\infty \frac{e^{-sx}}{x} dx \right\}$$

Hints for the proof

1. The Laplace transforms $\{g^{(n)}(s)\}_{n\geq 1}$ of $\{Z^{(n)}\}_{n\geq 1}$ satisfy a highly non-linear system of recursive identities

$$1 - g^{(n)}(s\, a_{n-1}) = (1 - g_{n-1}(s))e^{h^{(n-1)}(s)} \quad \forall n \geq 2\,.$$

where

$$a_n = d^{(n+1)}/d^{(n)}, \quad h^{(n)}(s) = \mathbb{E}\left(e^{-sZ^{(n)}}\chi_{1\leq Z^{(n)}<a_n}\right)$$

1. One observes that a family of fixed points is given by

$$g_c^{(\infty)}(s) = 1 - \exp\left(-c \int_1^\infty dx \, \frac{e^{-sx}}{x}\right), \quad c \in (0,1]$$

2. One then *proves* that there exists a non-negative measure $dt^{(n)}(x)$ such that

$$g^{(n)}(s) = 1 - \exp\left(-\int_1^\infty dt^{(n)}(x) \, \frac{e^{-sx}}{x}\right)$$

3. The recursive identitities for the $\{g^{(n)}(s)\}_{n\geq 1}$ become indentities for the measures $\{t^{(n)}\}_{n\geq 1}$ and one can *prove* that $t^{(n)} \to c \times Leb$ if

$$\lim_{s\downarrow 0} -s \frac{d}{ds} g^{(1)}(s)/(1 - g^{(1)}(s)) = c(\in (0,1])$$

HCP Universality and generalizations

- Previous result $\Rightarrow$ *Universality Classes*.
- One can generalize to $Q = \text{Ren}(\mu, \nu)$: scaling for the position of the first vacancy + same scaling for the rescaled domain length.
- Same method for triple coalescence (merging with left *and* right neighboring intervals) but *different* asymptotics.

East model vs HCP

Discussion about activation times and energy barriers suggest that, as $q \downarrow 0$, the East dynamics should be approximated by a suitable HCP such that ;

1. $d_n = 2^{n-1} + 1$;
2. time t_n^+ should corresponds to the end of the n-th coalescence ;
3. It remains to determine the appropriate rates $\lambda_n(d) \Leftrightarrow$ survival propability of a vacancy with domain d (of class n).
4. The rates are found solving a large deviation problem for the East dynamics. All what is needed is that the rates only weakly depend on q.

Main result

- The final result is that, on finite, large volume *independent* from q, the marginal for the East process and for the HCP above are close in $\|\cdot\|_{\mathrm{TV}}$ for $q \downarrow 0$.

- Staircase behavior and aging follow at once from the scaling limit of the HCP.

References

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