Limit shapes in the double Ising model

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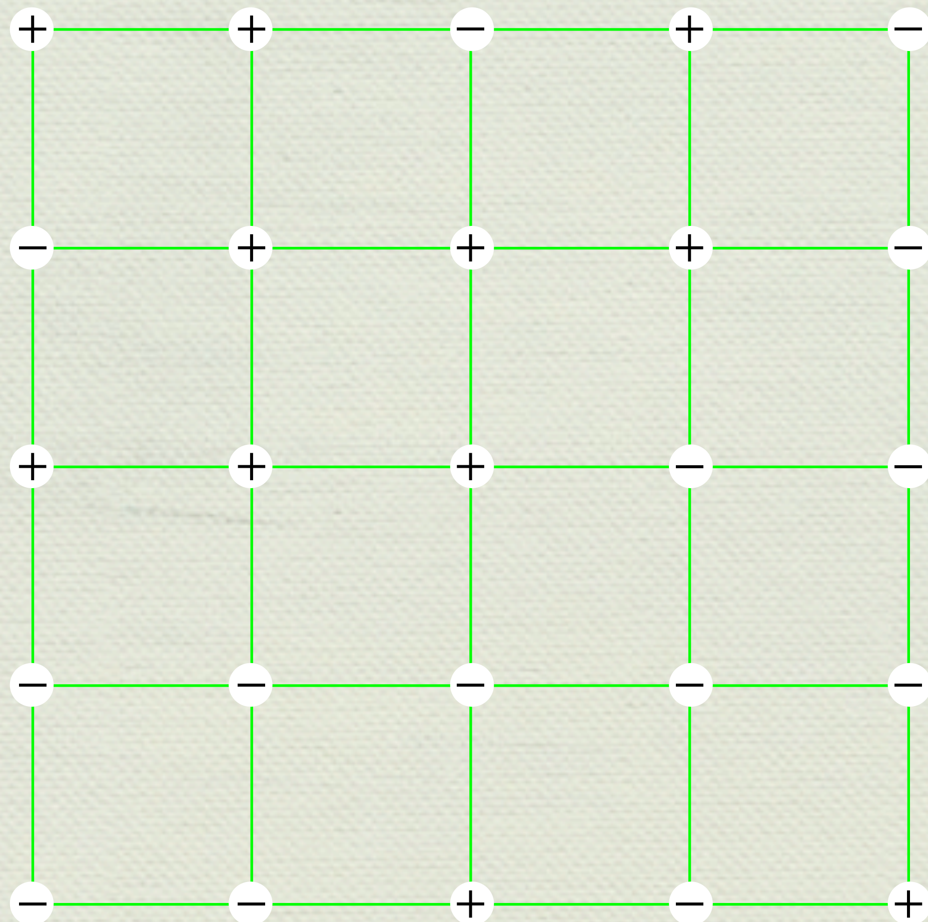
inspired by discussions with

Nike Sun, Robin Pemantle, David Wilson, Peter Winkler

Ising model



$$\Omega = \{+, -\}^G$$

for $\sigma \in \Omega$,

$$\text{Prob}(\sigma) = \frac{1}{Z} \exp\left(\beta \sum_{\text{edges } xy} \delta_{\sigma(x)=\sigma(y)} \right)$$

Ising characteristic polynomial $\quad t = e^{\beta}$

$$P(z,w) = (t^2+1)^2 + t(t^2-1)(z + \frac{1}{z} + w + \frac{1}{w})$$

[Onsager] Free energy: $\quad F = \quad \frac{-1}{4\pi^2} \iint_{|z|=|w|=1} \log P(z,w)\, \frac{dz}{z}\, \frac{dw}{w}$

[Fisher] Energy-energy correlations:

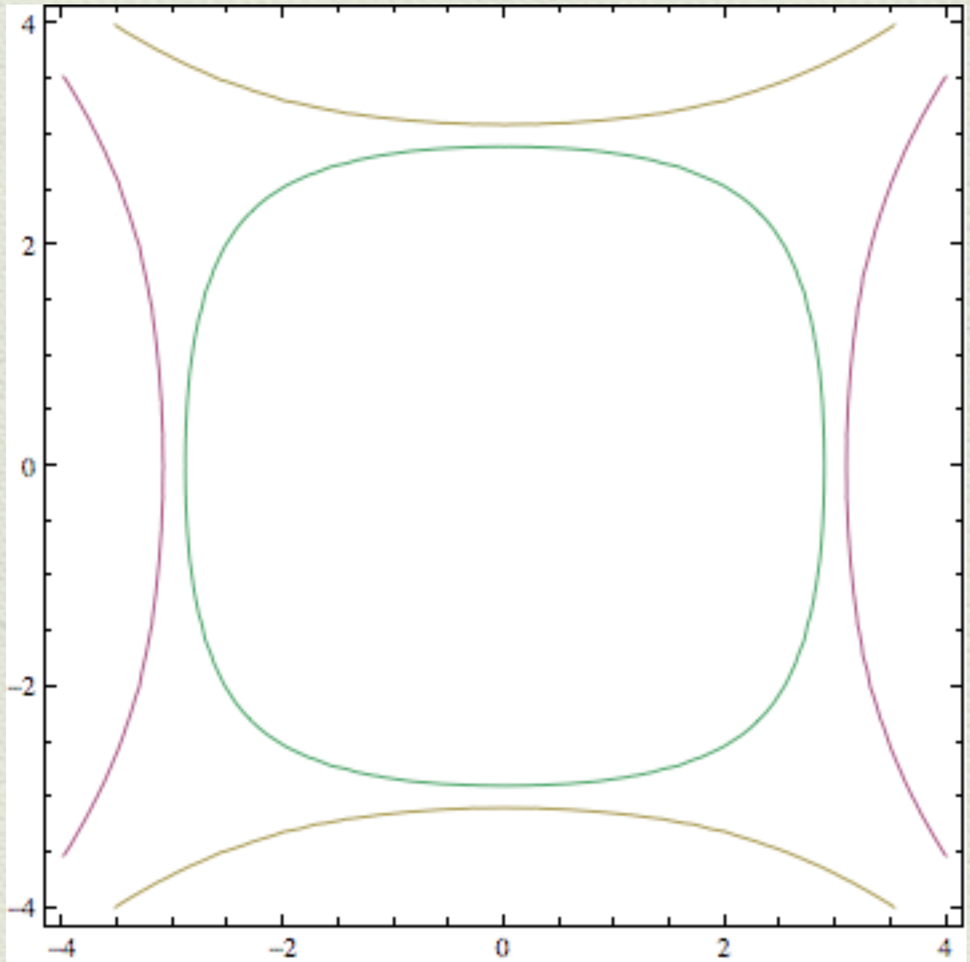
obtained from Fourier coefficients of $1/P(z,w)$.

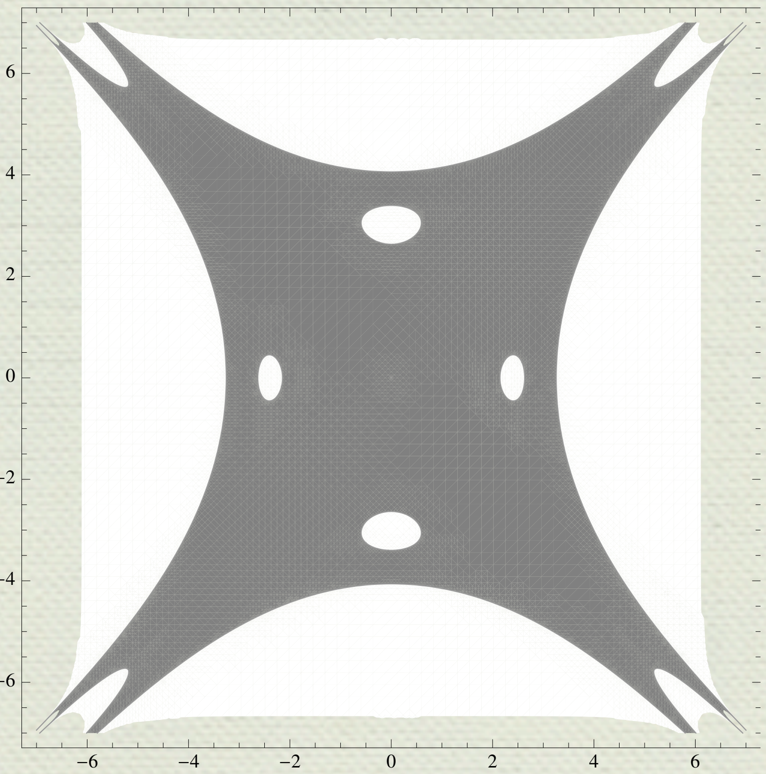
[Z. Li (2011)] Critical β: when P has roots on $\mathbb{T}^2$.

(when $\frac{(t^2+1)^2}{t(t^2-1)} = 4$)

Note also: $\quad \frac{(t^2+1)^2}{t(t^2-1)} \geq 4$

Amoeba of P $\{(\log|z|, \log|w|) \ : \ P(z,w) = 0\}$





More complicated graph
$\Longrightarrow$ more features

What is the probabilistic meaning of the "rest" of P?

compare to:

Bipartite dimer model:

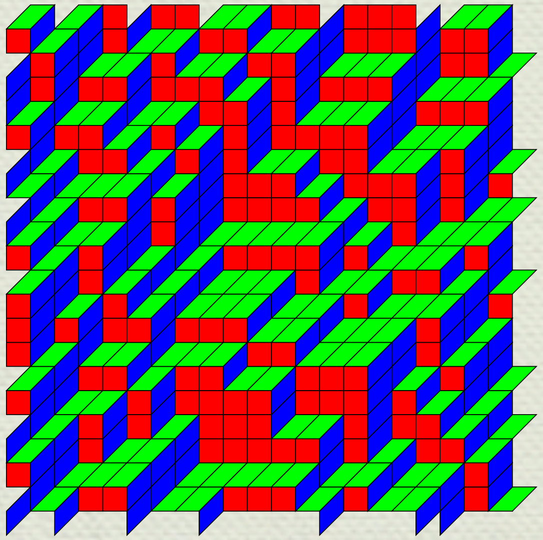
"Translates" of P are characteristic polynomials

for other Gibbs states (gradient measures).

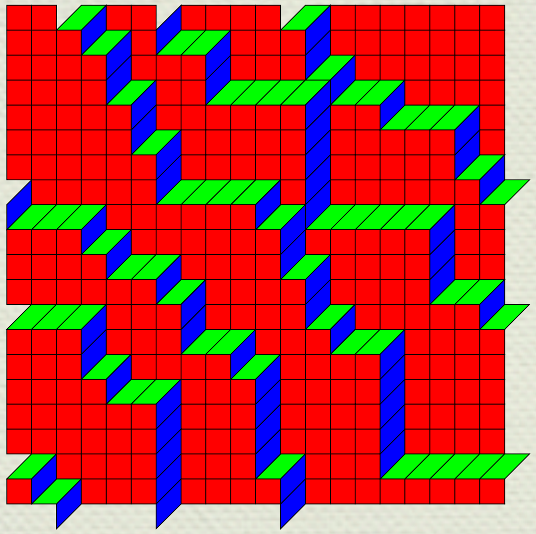
$$P(z,w)=\sum_{j,k}C_{j,k}z^jw^k$$

Translate $(-x,-y)$ to origin:

$$P(e^xz,e^yw)=\sum_{j,k}C_{j,k}e^{jx+ky}z^jw^k$$



$1+z+w$



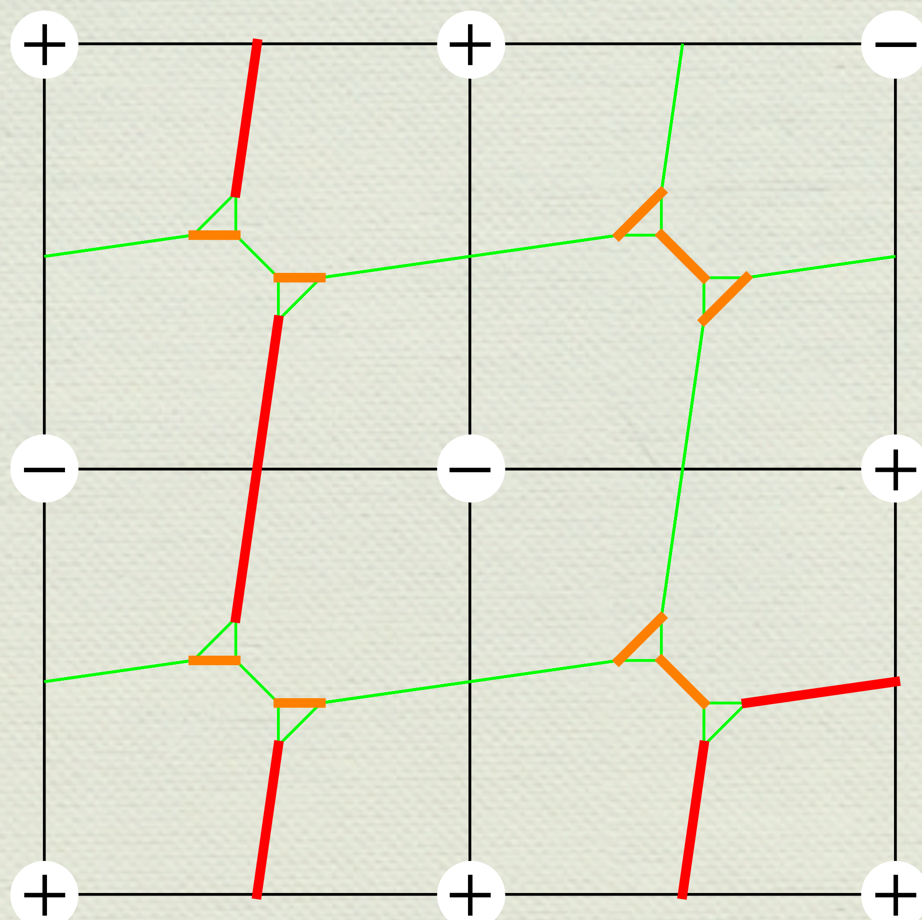
$1+\frac{2}{3}z+\frac{2}{3}w$

Ising model:

do translations of P correspond to other measures? Yes!

…but these are not (Ising) Gibbs measures

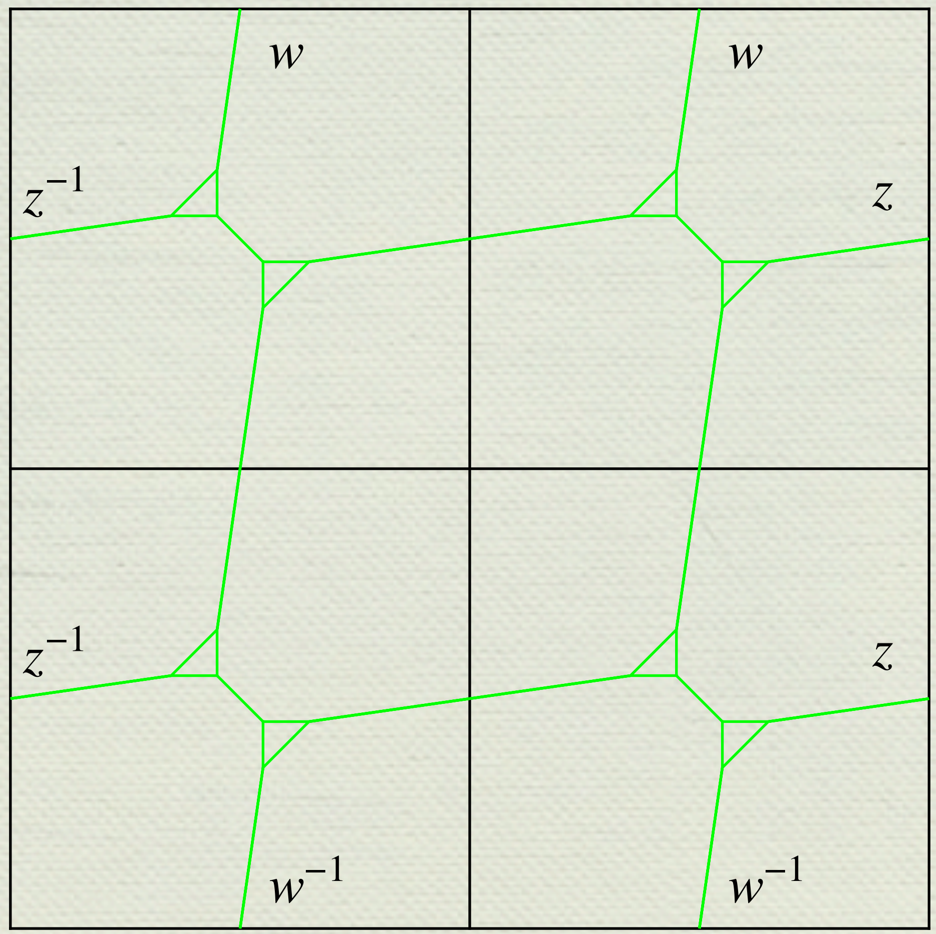
Dimer/Ising correspondence (Kasteleyn/Fisher)



long edges have weight e^{β}

short edges have weight 1.

Kasteleyn matrix



G is a graph on the torus;

$K(z,w)$: oriented adjacency matrix of green graph with certain orientations and weights.

Theorem [Kasteleyn]

$P(z,w) = \det K(z,w)$ is a weighted, signed sum of double-dimer covers:

$$P(z,w) = \sum_{i,j} C_{i,j} z^i w^j$$

$C_{i,j}$ is a weighted, signed sum of double-dimer configurations containing (oriented) loops with total homology class (i,j).

(signs are due to topologically nontrivial loops with "wrong" parity)

Also: not every double dimer cover comes from an Ising configuration

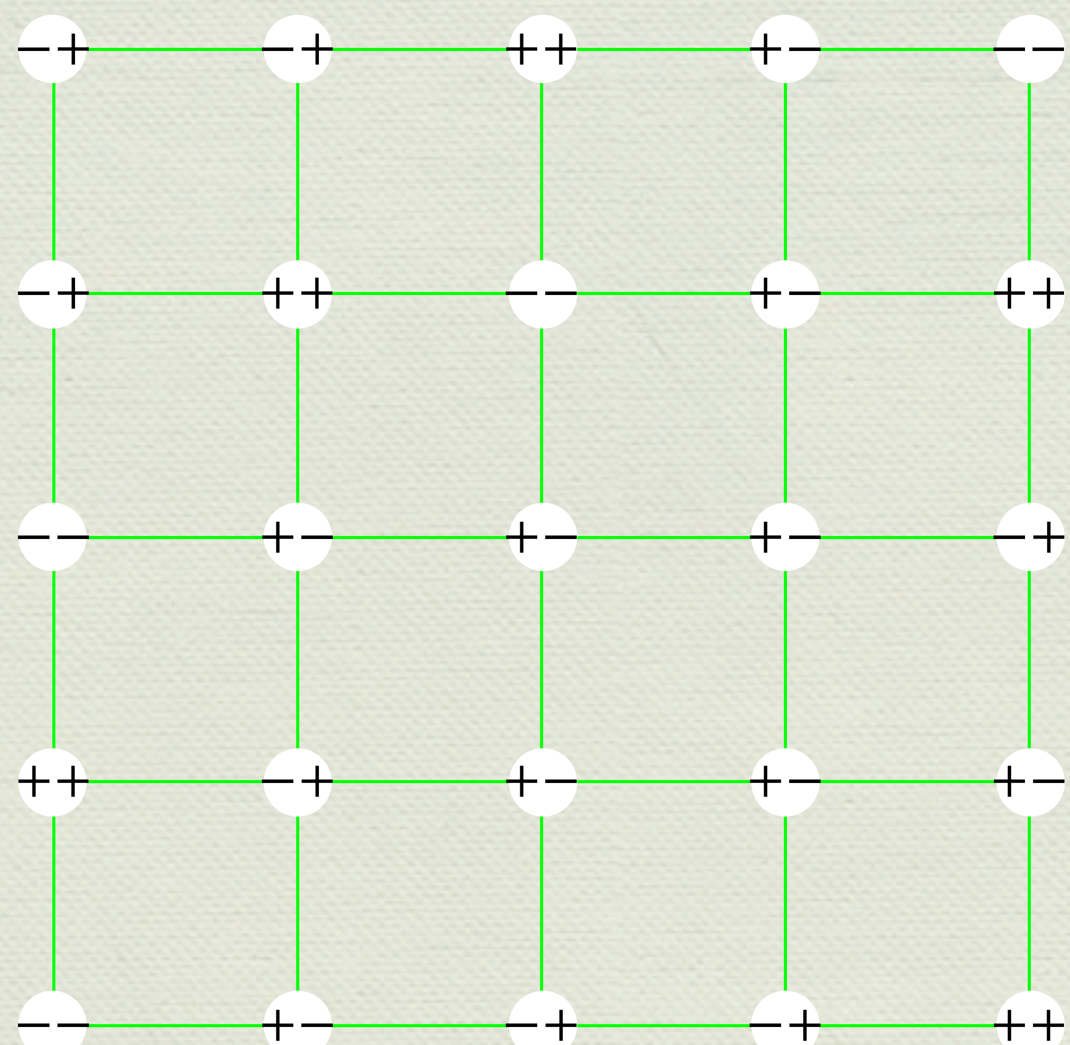
... there may be a spin change when going horizontally or vertically around $\mathbb{T}^2$.

Let (σ_x, σ_y) be this spin change.

Idea: write $C_{i,j}$ as a sum of positive terms.

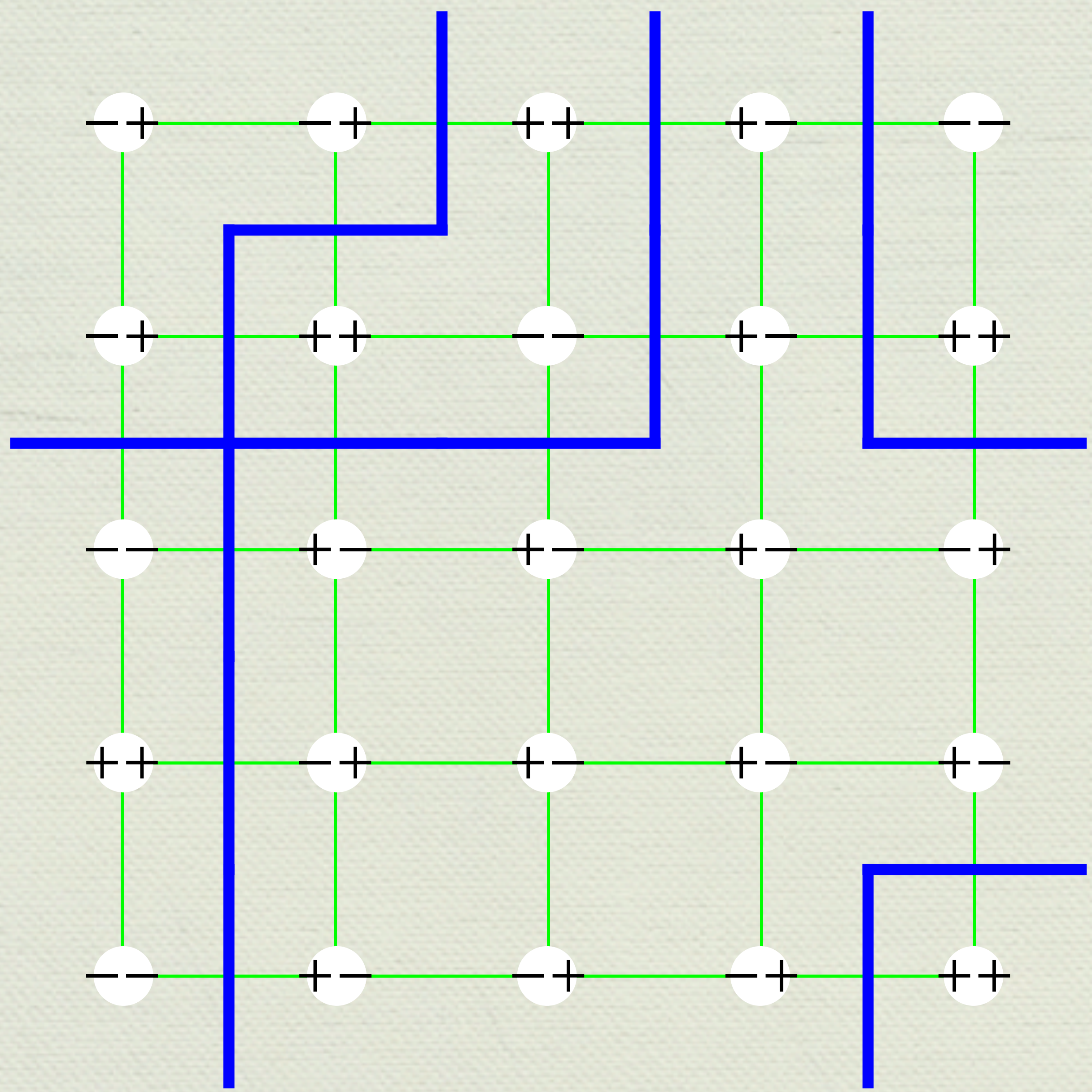
Another interpretation of $C_{j,k}$

Double Ising model

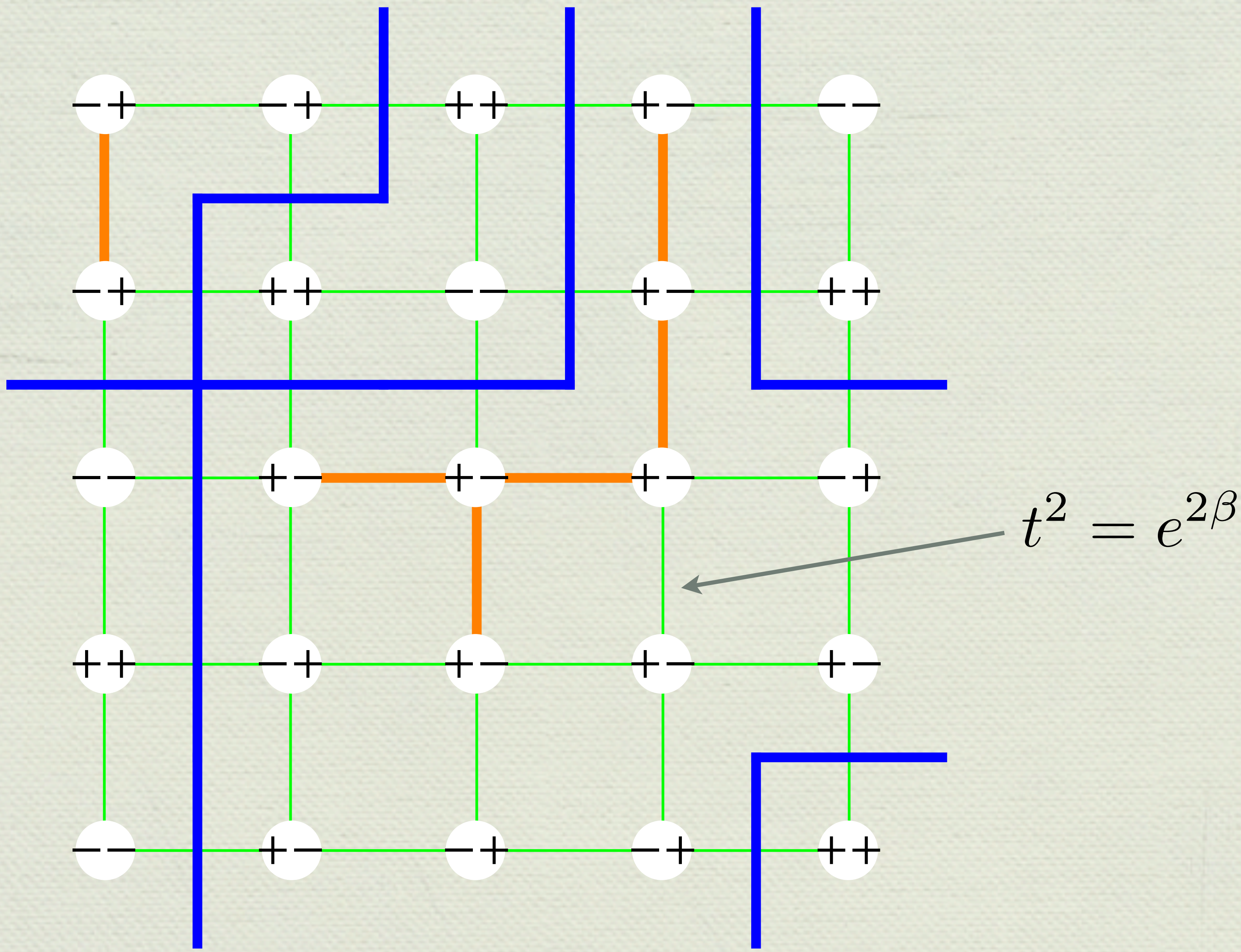


Two independent Ising models.

XOR-spin domain boundaries



Random cluster model on XOR spin domains

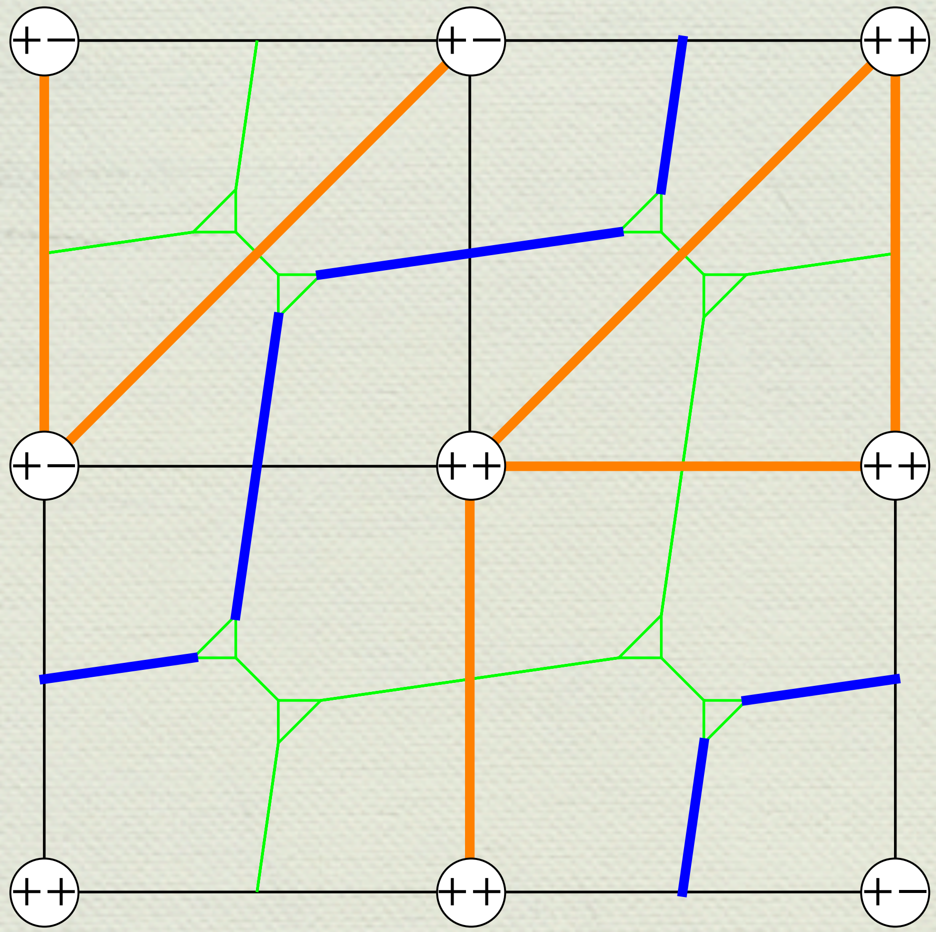


Choose edges joining like double-spins with probability $\frac{t^2}{1+t^2}$.

For G on a torus:

$C_{j,k}$: • the XOR domain boundaries (blue) have homology (j, k)

• the FK clusters (orange) percolate around the annuli



$$(j, k) = (1, 1)$$

$$(\sigma_x, \sigma_y) = (+-, +-)$$

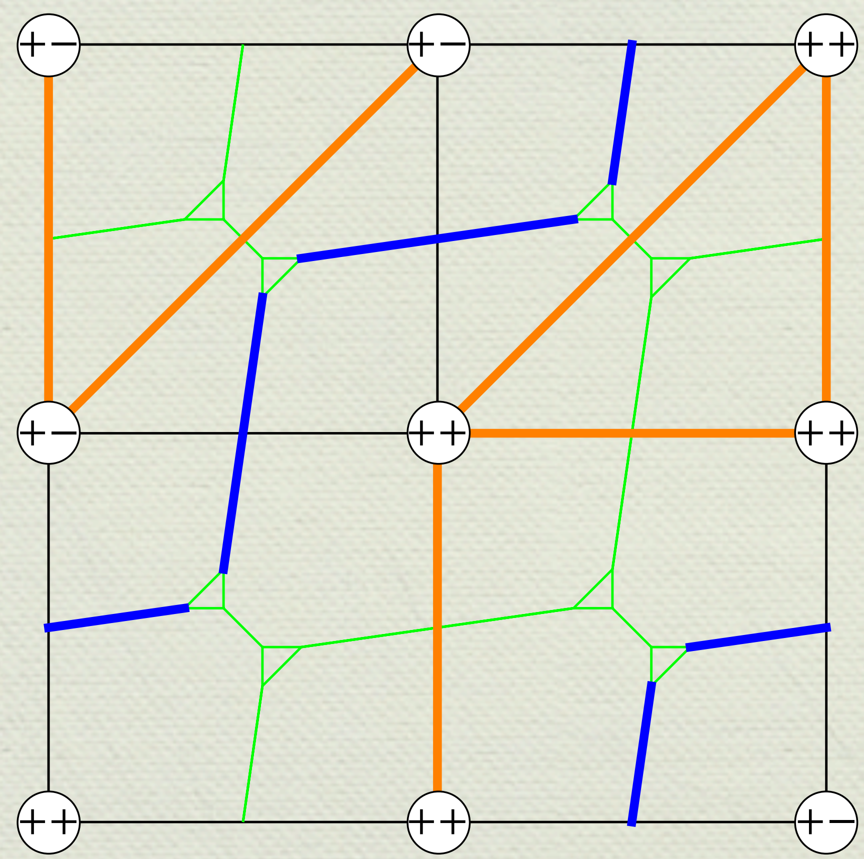
Theorem: $C_{i,j} = \sum_\gamma wt(\gamma) Z_{\mathrm{FK}_2}(\gamma^c)$.

where γ runs over possible XOR-spin domain boundaries (blue)
with homology (i,j) (and edge weights e^β) and

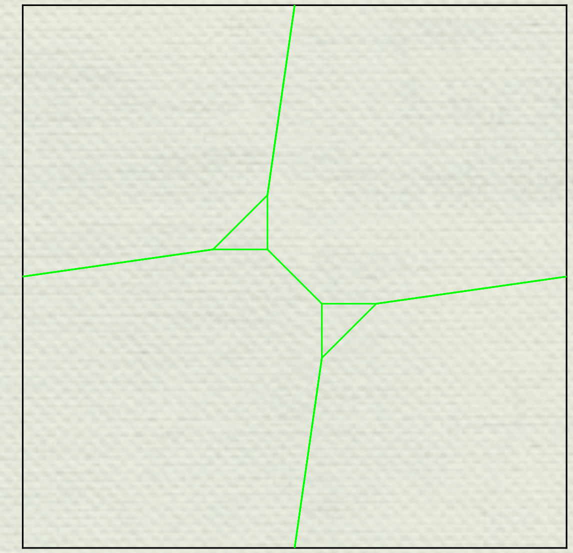
$Z_{\mathrm{FK}_2}(\gamma^c)$ =partition sum of FK_2 model on complementary domains,

conditioned to percolate on oriented annuli. (orange)

(edge weights $e^{2\beta} - 1$)

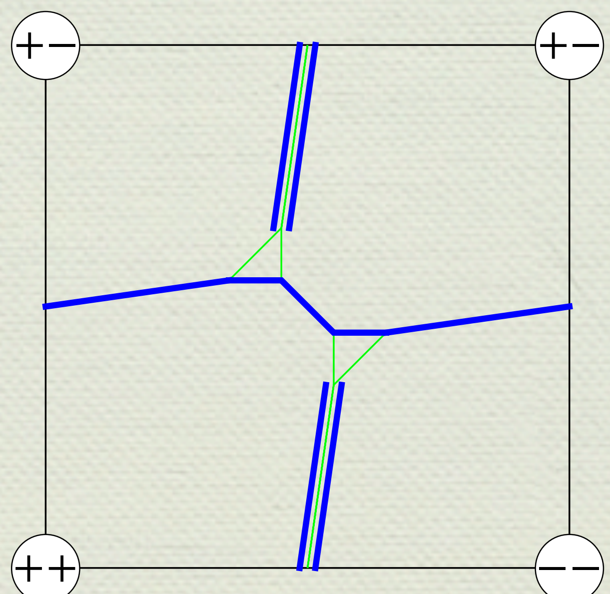


Simple example: 1×1 torus

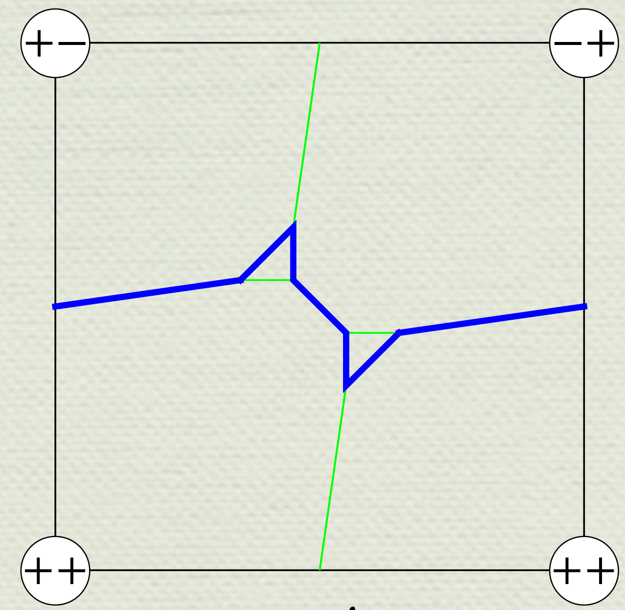


$$P(z,w) = (t^2+1)^2 + (t^3-t)(z+1/z+w+1/w)$$

$$C_{1,0} = t(t^2-1)$$

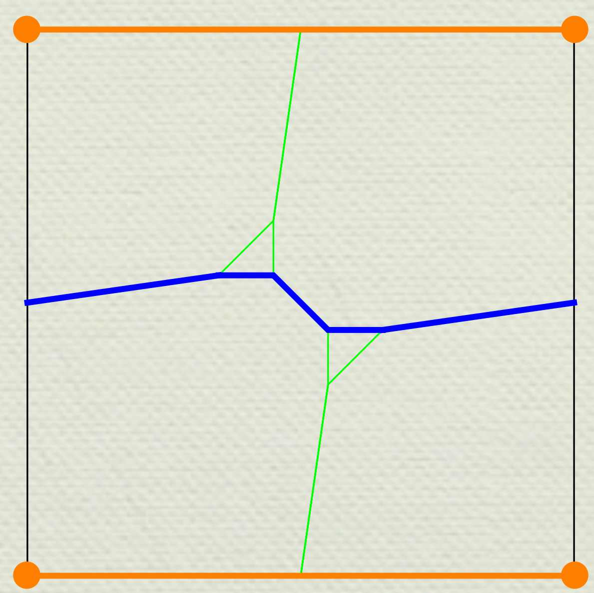


t^3



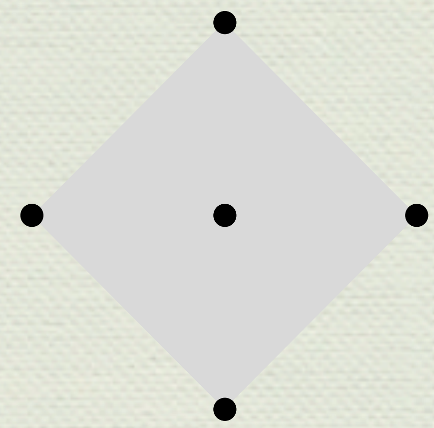
$-t$

$C_{1,0}$ = weight of:



"Banded states" for the double Ising model

For each $(s, t) \in N$ there is a measure $\mu_{s,t}$ on Ising/FK configurations.

$N =$  $=$ Newton polygon of P

$\mu_{s,t}$ has FK-components with horiz./vert. density s, t.

$\mu_{0,0} =$ unconstrained double Ising model.

$\mu_{1,0} = \mu_{-1,0} =$ all vertical edges.

$\mu_{s,t}$

oops, this is a figure from a triangular Ising model

"Surface tension" (entropy) $\sigma_{s,t}$ of measure $\mu_{s,t}$

To compute $C_{\lfloor ns \rfloor, \lfloor nt \rfloor}$ for the $n \times n$ torus, use

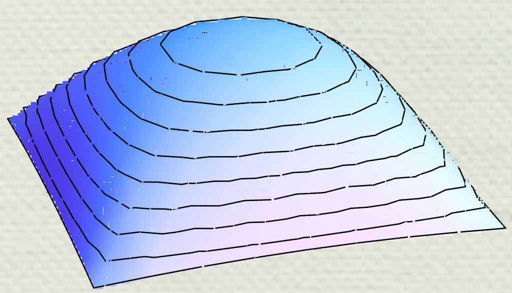
$$P_{n\times n}(z,w) = \prod_{\zeta^n = z, \eta^n = w} P_{1\times 1}(\zeta, \eta)$$

We define the **surface tension** $\sigma_{s,t} := \lim_{n\to\infty} \frac{1}{n^2} \log C_{\lfloor ns \rfloor, \lfloor nt \rfloor}$.

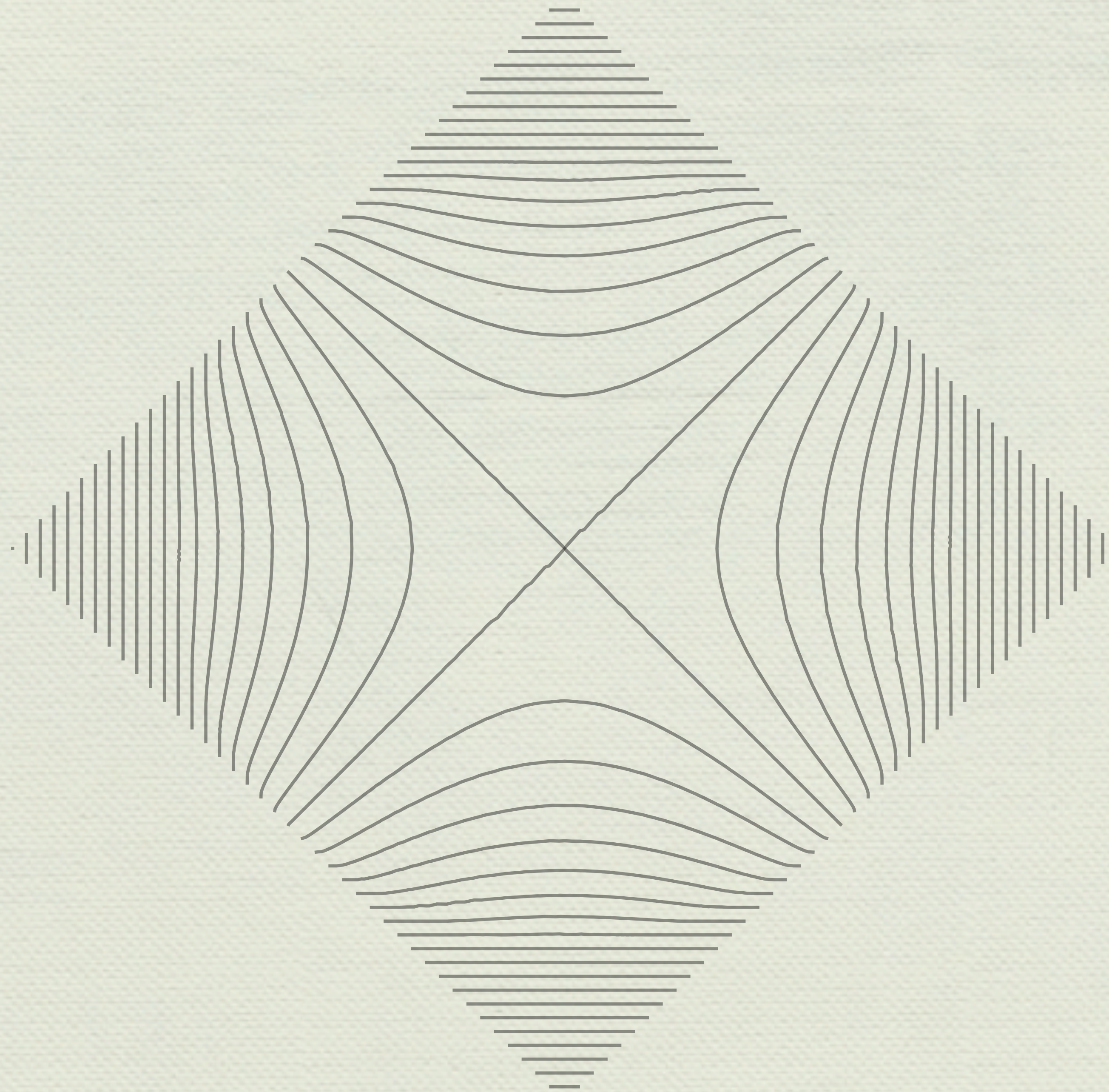
Lemma [KOS]: $\sigma(s,t) =$ Legendre dual of $R(x,y)$ where

$$R(x,y) = \frac{1}{(2\pi i)^2} \iint_{|z|=|w|=1} \log P(e^x z, e^y w) \, \frac{dz}{z} \, \frac{dw}{w}.$$

$R(x,y)$ is the "Ronkin function" of P.



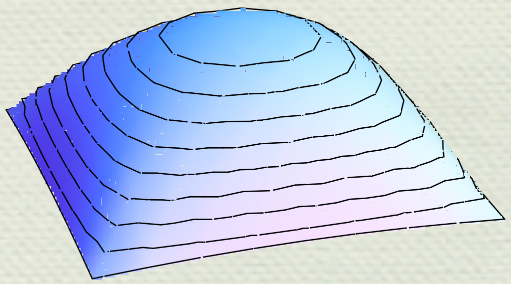
Surface tension

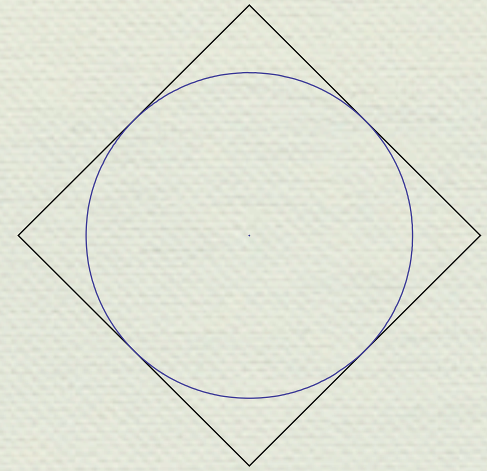


Limit shape for banded model

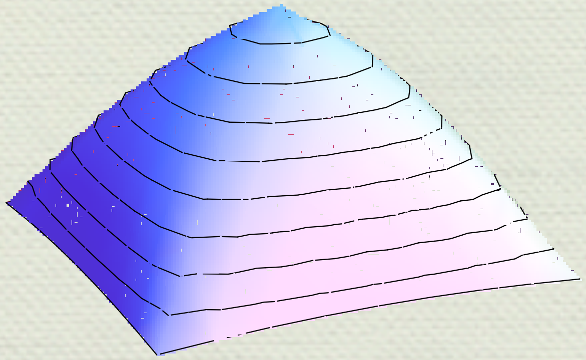
conditioned on certain boundary connections surface tensions

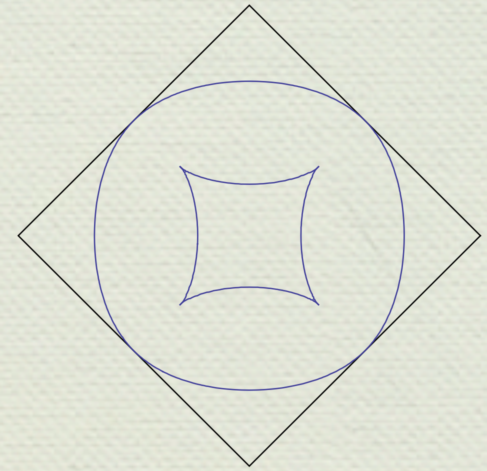
limit shape boundaries



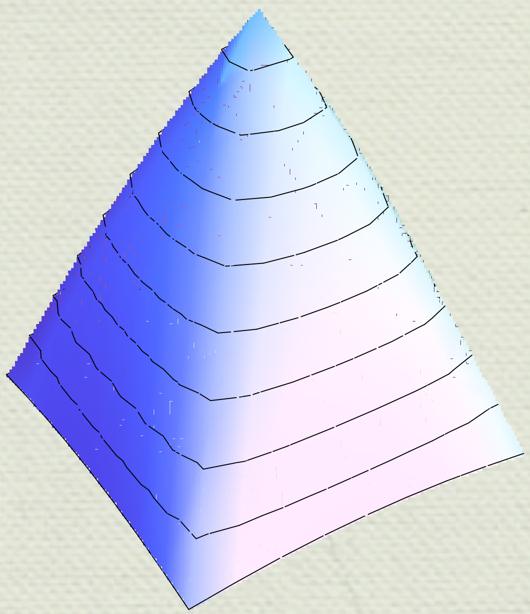


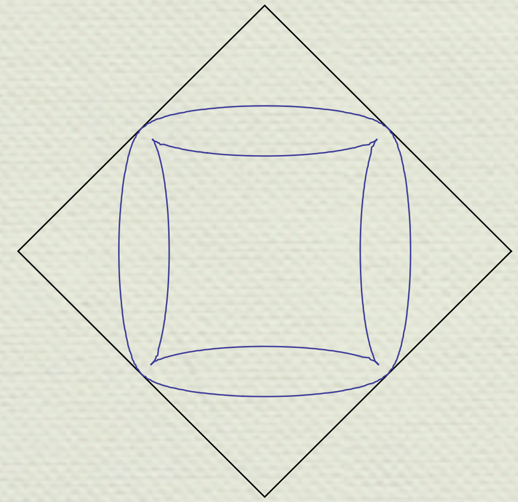
critical temperature



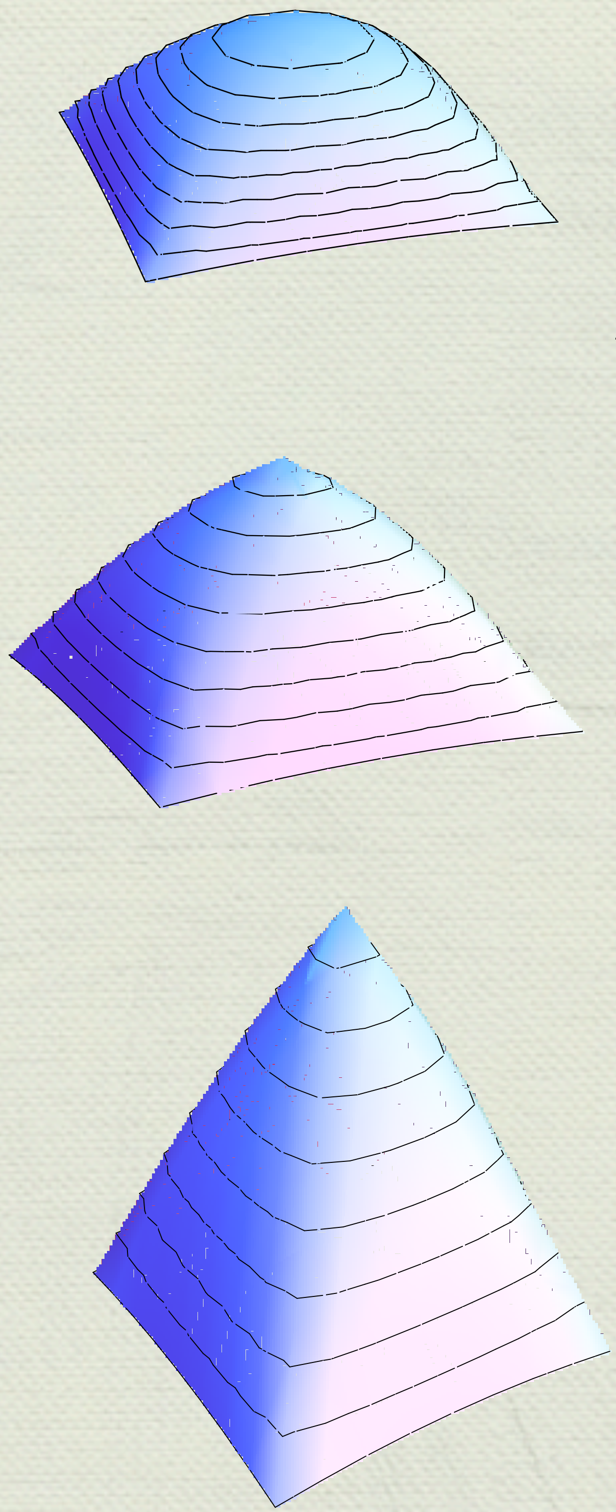


subcritical temperature

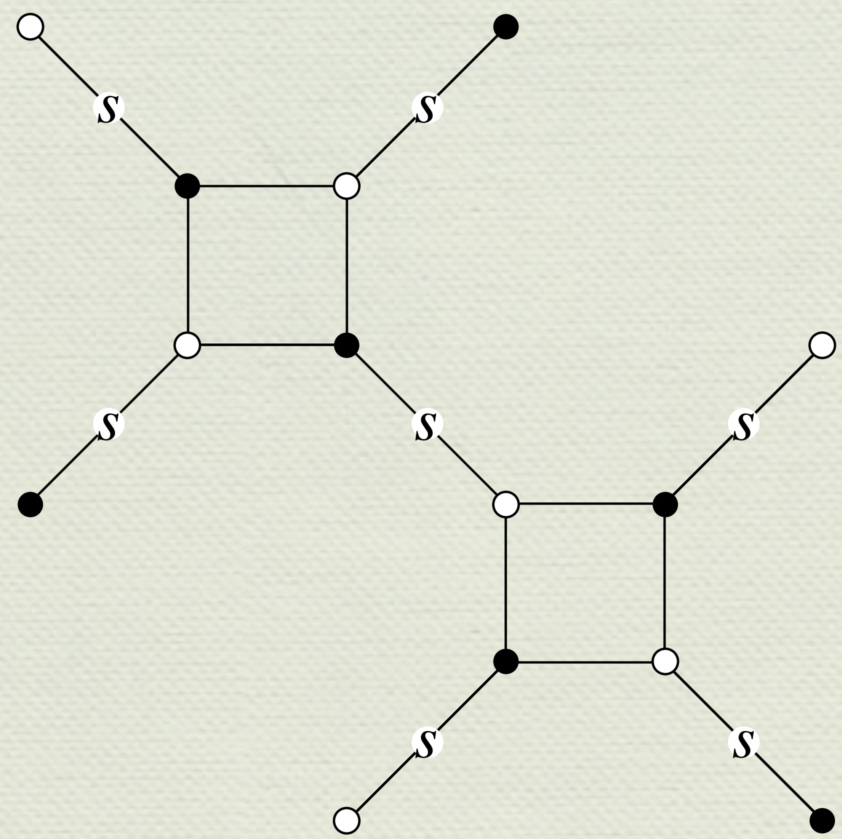




even more subcritical temperature



These surface tensions are identical
to those arising in the square-octagon dimer model



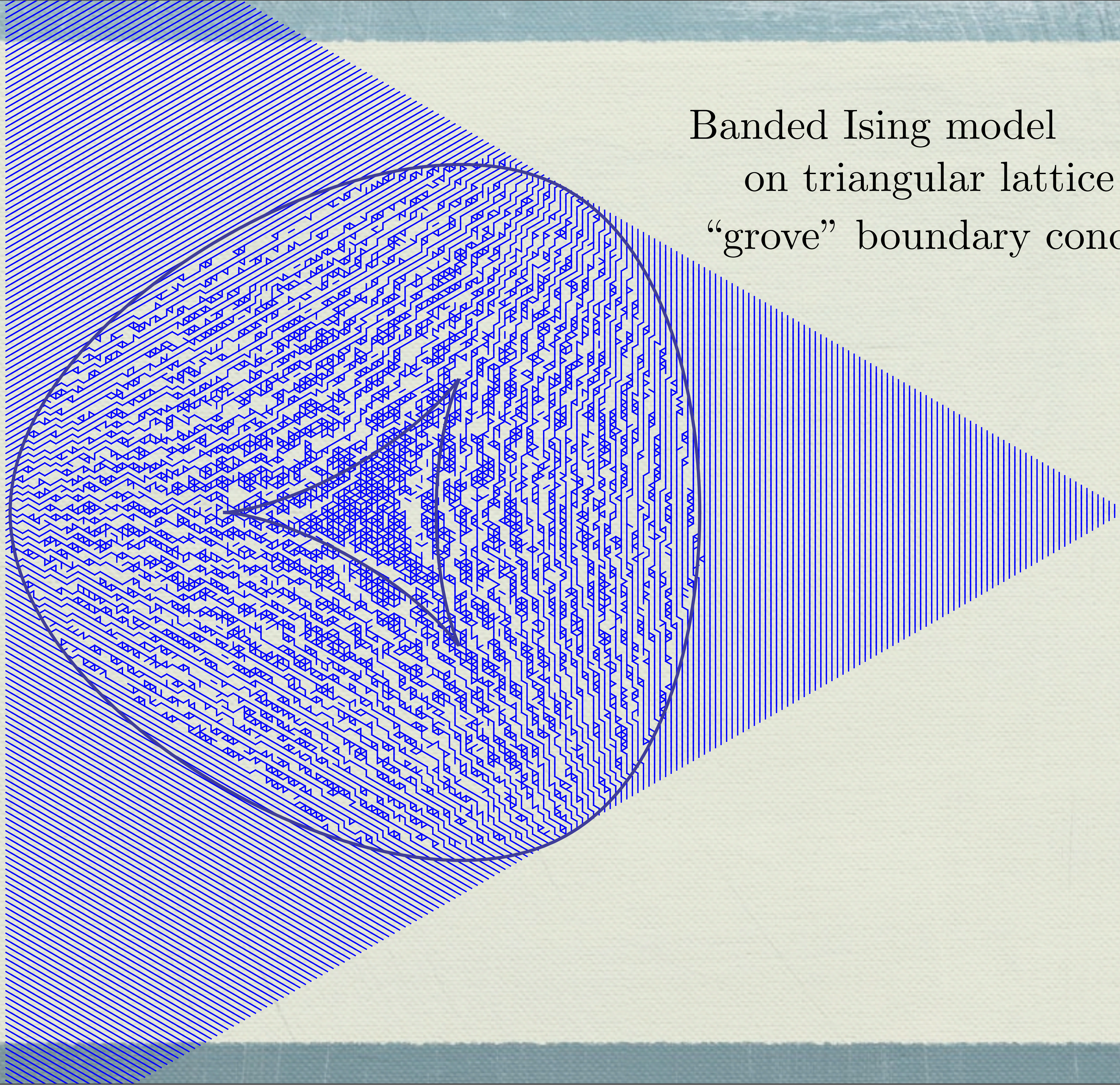
$$s = s(\beta)$$

Conjecture: For any periodic planar graph, the banded Ising model surface tension equals that arising from some bipartite dimer model.

Follows from:

Conjecture: The spectral curve $P(z, w) = 0$ in the ferromagnetic Ising model on any periodic planar graph is a simple Harnack curve.

Banded Ising model
on triangular lattice
“grove” boundary conditions



$$Q = 3098115099749559840207375698 41a^6 - 1858374937729039544359650269170a^5b - 1858374937729039544359650269170a^5c + 5883454153820320725807778237007a^4b^2 + 4334397195006546369711336315654a^4bc + 5883454153820320725807778237007a^4c^2 - 8669781452132474330937731075356a^3b^3 - 7427079315358238395356762728212a^3b^2c - 7427079315358238395356762728212a^3bc^2 - 8669781452132474330937731075356a^3c^3 + 5883454153820320725807778237007a^2b^4 - 7427079315358238395356762728212a^2b^3c + 32797543284281898673568730387594a^2b^2c^2 - 7427079315358238395356762728212a^2bc^3 + 5883454153820320725807778237007a^2c^4 - 1858374937729039544359650269170ab^5 + 4334397195006546369711336315654ab^4c - 7427079315358238395356762728212ab^3c^2 - 7427079315358238395356762728212ab^2c^3 + 4334397195006546369711336315654abc^4 - 1858374937729039544359650269170ac^5 + 309811509974955984020737569841b^6 - 1858374937729039544359650269170b^5c + 5883454153820320725807778237007b^4c^2 - 8669781452132474330937731075356b^3c^3 + 5883454153820320725807778237007b^2c^4 - 1858374937729039544359650269170bc^5 + 309811509974955984020737569841c^6$$