Conformal Invariance of double-dimer loops

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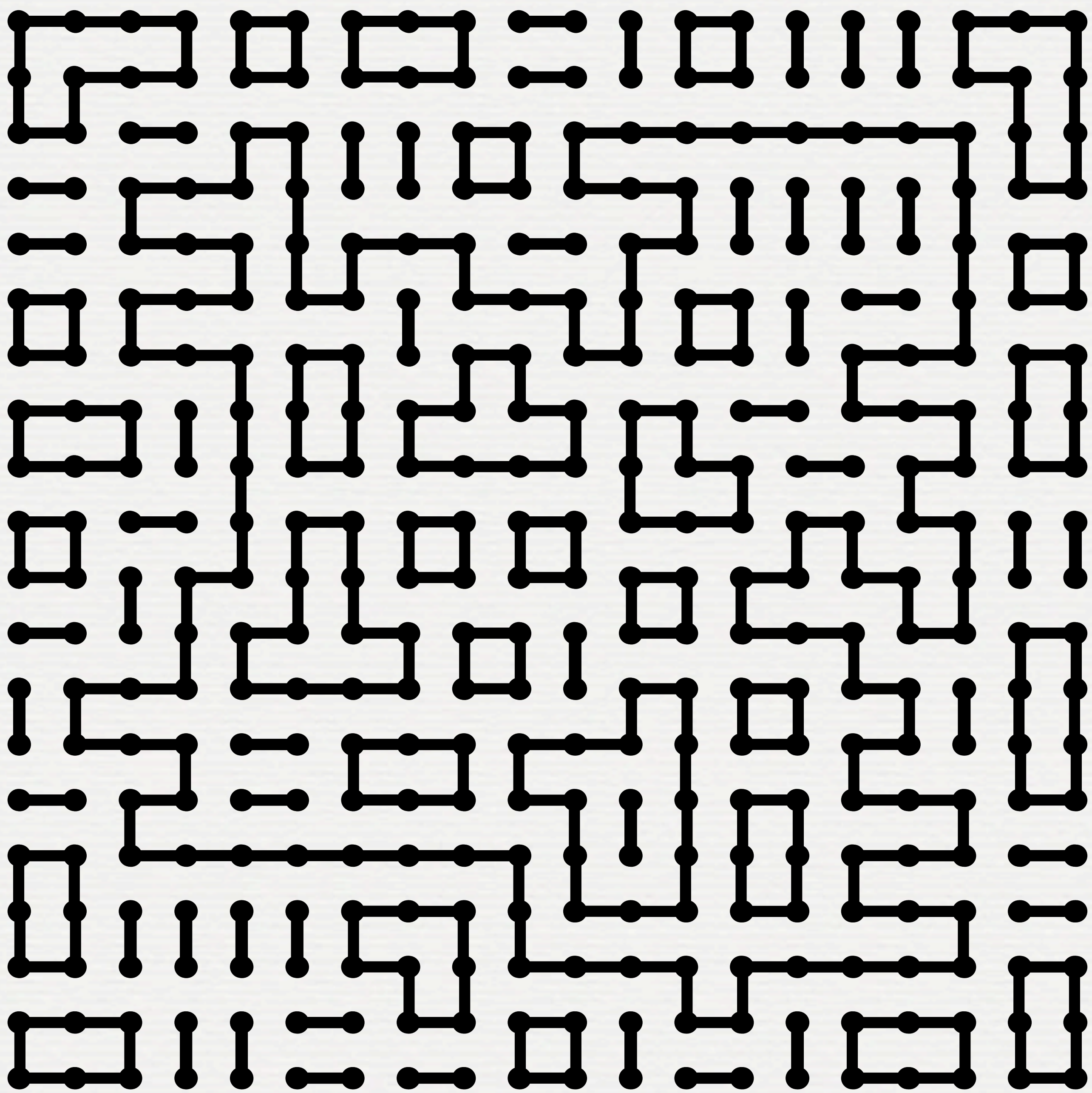
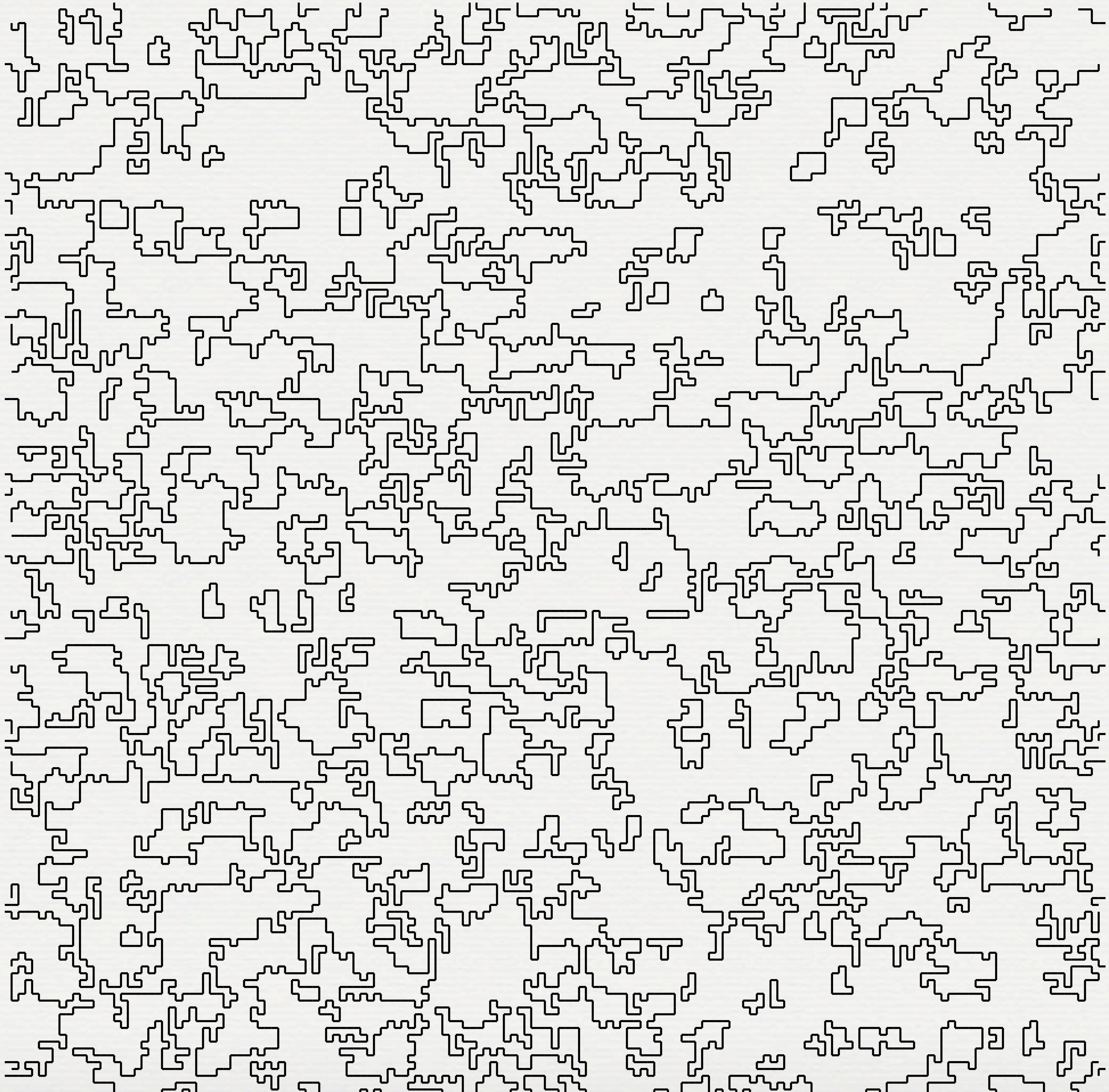
Double-Dimer model

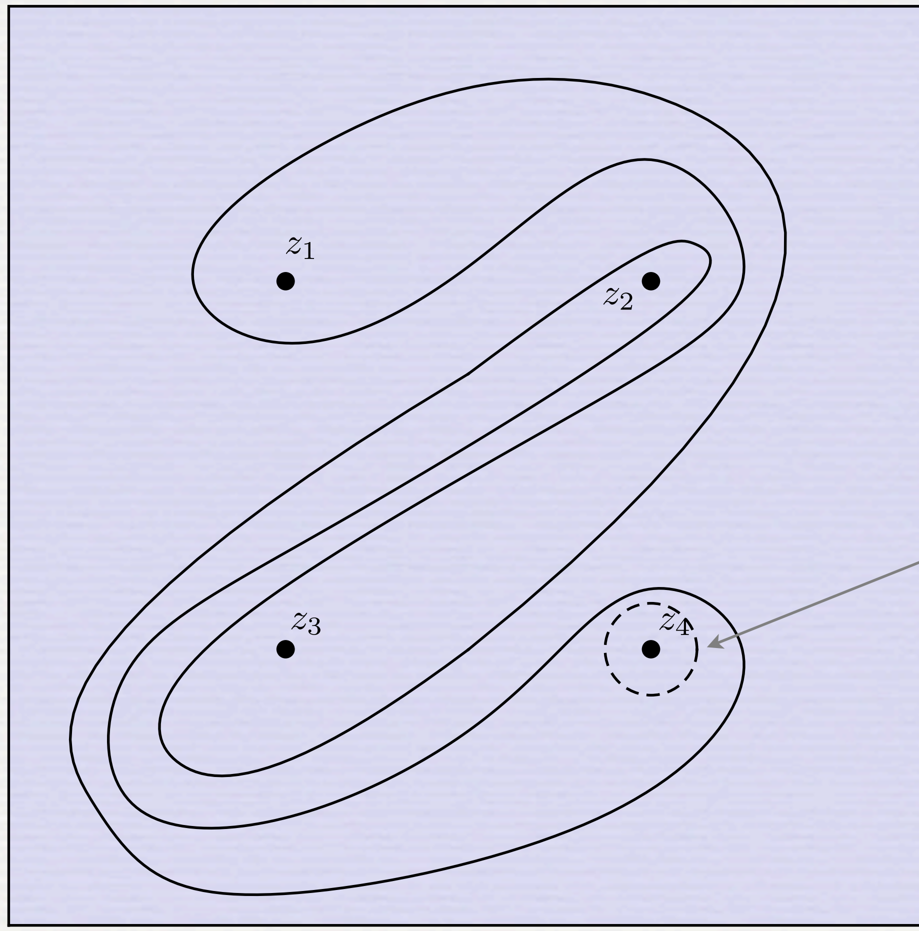


Two independent uniform random dimer coverings

Long loops.



On a surface, a **finite lamination** is an isotopy class of pairwise disjoint non-contractible, non-peripheral simple closed curves.



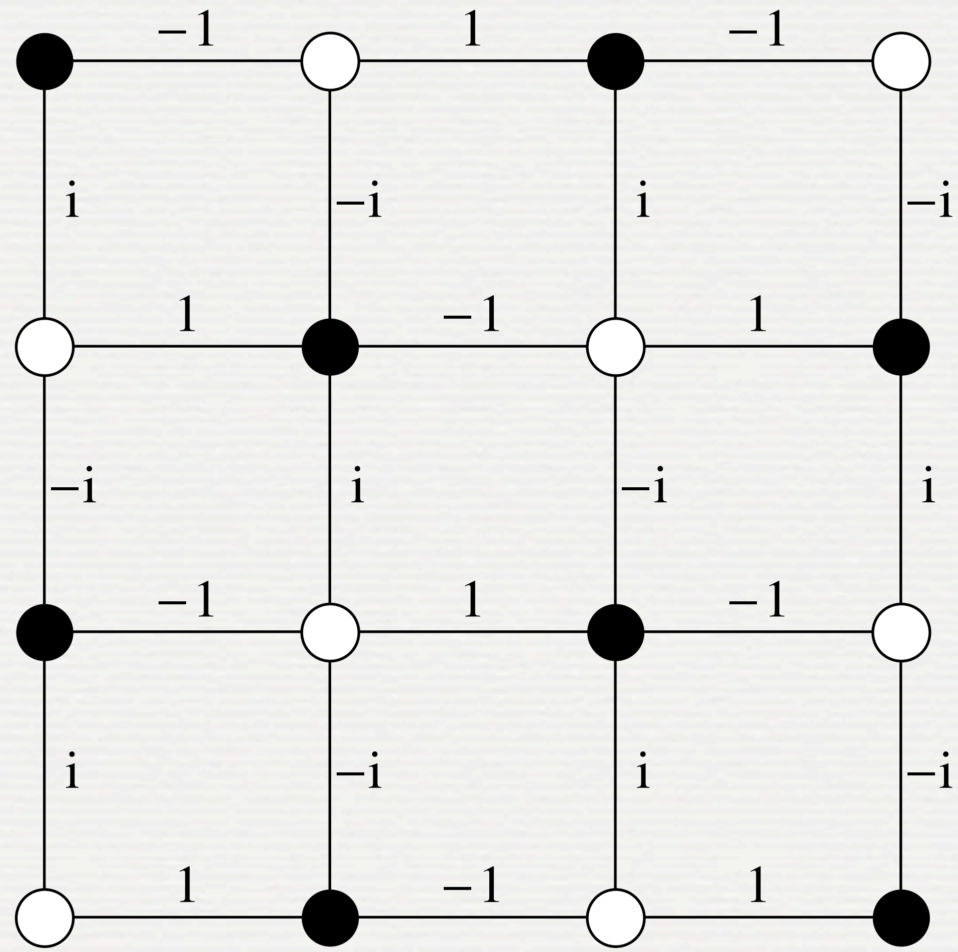
peripheral
= homotopic to a puncture

Theorem: Let $U_\epsilon = U \cap \epsilon\mathbb{Z}^2$.

Let $z_1, \ldots, z_k$ points in U. For each finite lamination L in $U \setminus \{z_1, \ldots, z_k\}$, $\Pr_\epsilon(L)$ converges and depends only on the conformal type of $U \setminus \{z_1, \ldots, z_k\}$.

(Ignore peripheral and contractible loops.)

Single dimer Kasteleyn matrix for $\mathbb{Z}^2$

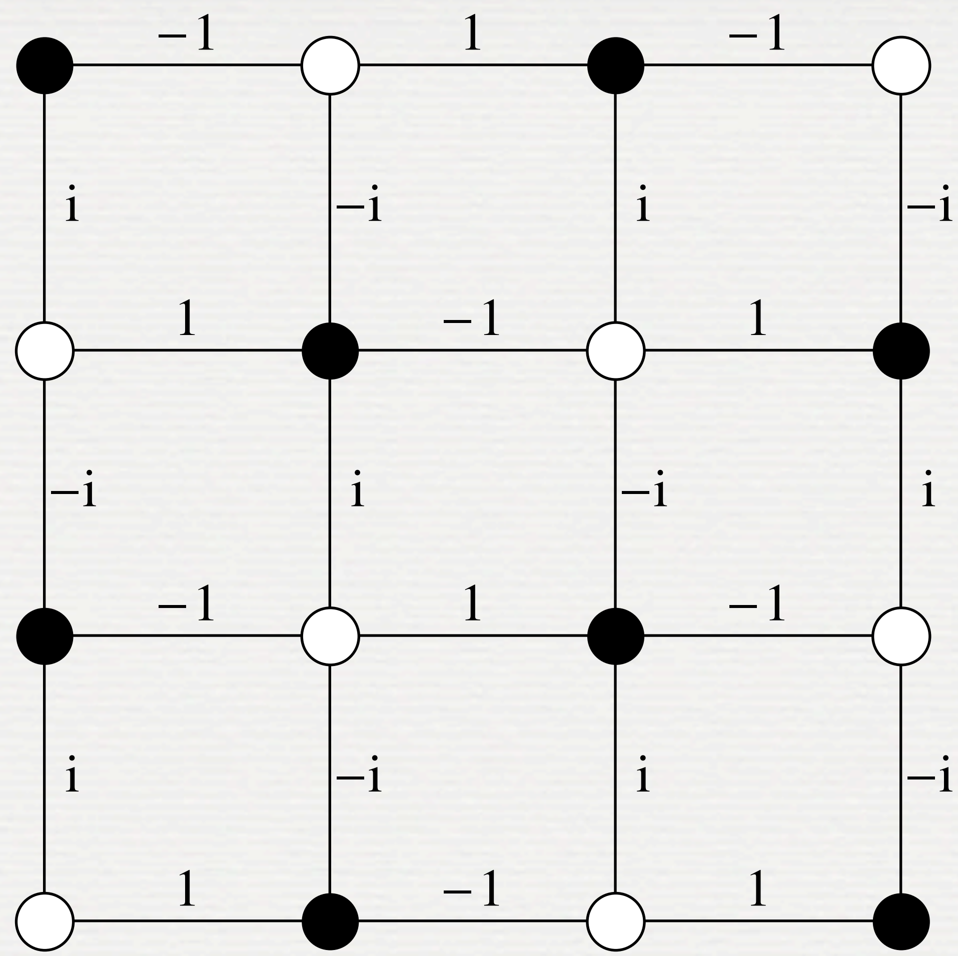


Theorem[Kasteleyn] Let $K : \mathbb{C}^W \to \mathbb{C}^B$ as above.

$K = (K_{w,b})$ where $K_{w,b} = 0$ if w, b are not adjacent and otherwise

$K_{w,b} = \{1, i, -1, -i\}$ according to direction.

Then $|\det K|$ is the number of dimer covers.



$$Kf(w) = f(w+1) - f(w-1) + i(f(w+i) - f(w-i))$$

$$= \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(w)$$

$$= \frac{\partial}{\partial \bar{z}} f(w)$$

A function in the kernel of K is *discrete analytic.*

Double dimer model

$$\text{Let } \mathbb{K} = \begin{pmatrix} 0 & K \\ K^t & 0 \end{pmatrix}. \qquad \text{(indexed by } \textit{all} \text{ vertices).}$$

Then $\det \mathbb{K}$ is the partition function of double-dimer configurations.

Let $\Omega(G)$ be the set of "double-dimer configurations":

coverings of G with even-length loops and doubled edges.

Each configuration in Ω with k nontrivial loops comes

from 2^k pairs of dimer covers.

Double dimer model

Now introduce quaternionic (instead of positive real) edge weights.

$$q = a_0 + a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = \begin{pmatrix} a_0 + a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & a_0 - a_1 i \end{pmatrix}.$$

$$q^* = a_0 - a_1\hat{\imath} - a_2\hat{\jmath} - a_3\hat{k}$$

The weight of $\omega \in \Omega$ is now defined to be

$$\prod_{\text{cycles}} (m + m^*)$$

where m is the product of the edge weights around the cycle.

Note $(q_1 q_2 \ldots q_k)^* = q_k^* \ldots q_1^*$.

Doubled edges count qq^*.

Theorem: $$\operatorname{Qdet} \mathbb{K} = \sum_{\Omega} \prod_{\text{cycles}} (m + m^*).$$

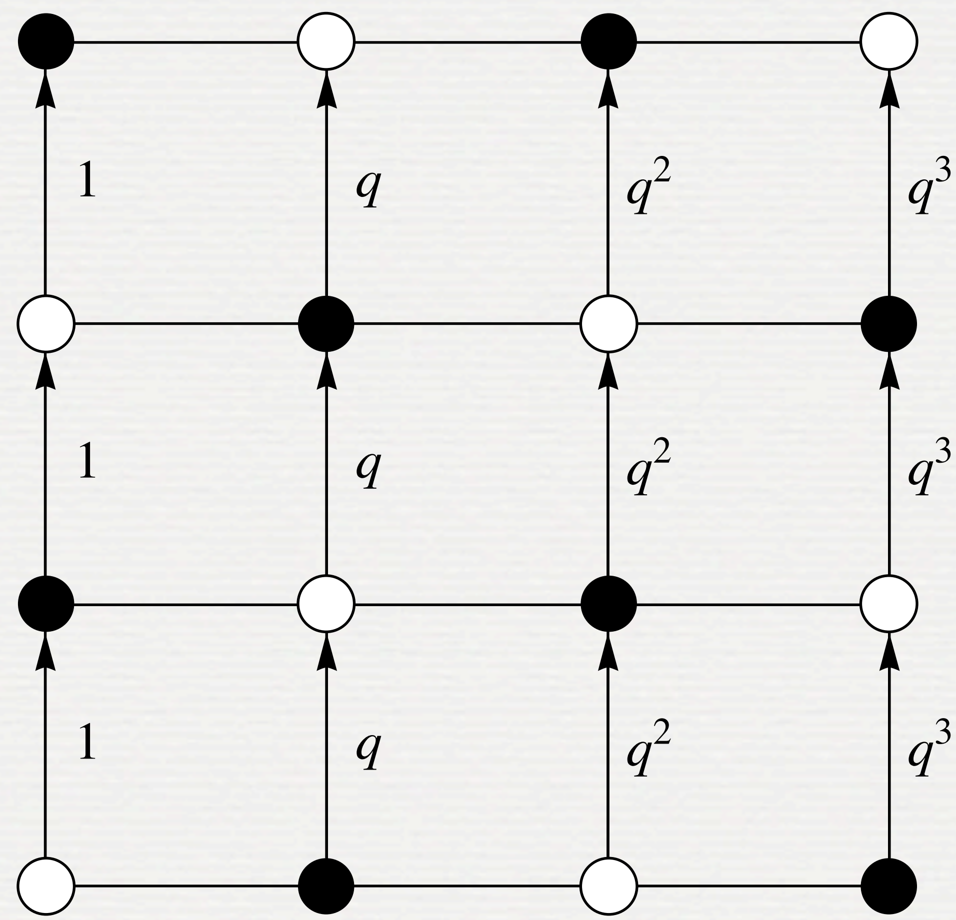
Here Qdet is the quaternion determinant [Dyson].

It is defined for a quaternion-Hermitian matrix by

$$\operatorname{Qdet} M = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{\substack{\text{cycles } c \text{ of } \sigma \\ \text{sorted and in order}}} m(c)$$

Theorem [Mehta]. $\operatorname{Qdet} \mathbb{K}_{n\times n} = \sqrt{\det \widetilde{\mathbb{K}}_{2n\times 2n}}$, where $\tilde{\mathbb{K}}$ is the matrix obtained by replacing each quaternion with its 2×2 matrix block.

Example: put weights q^x on north edges:

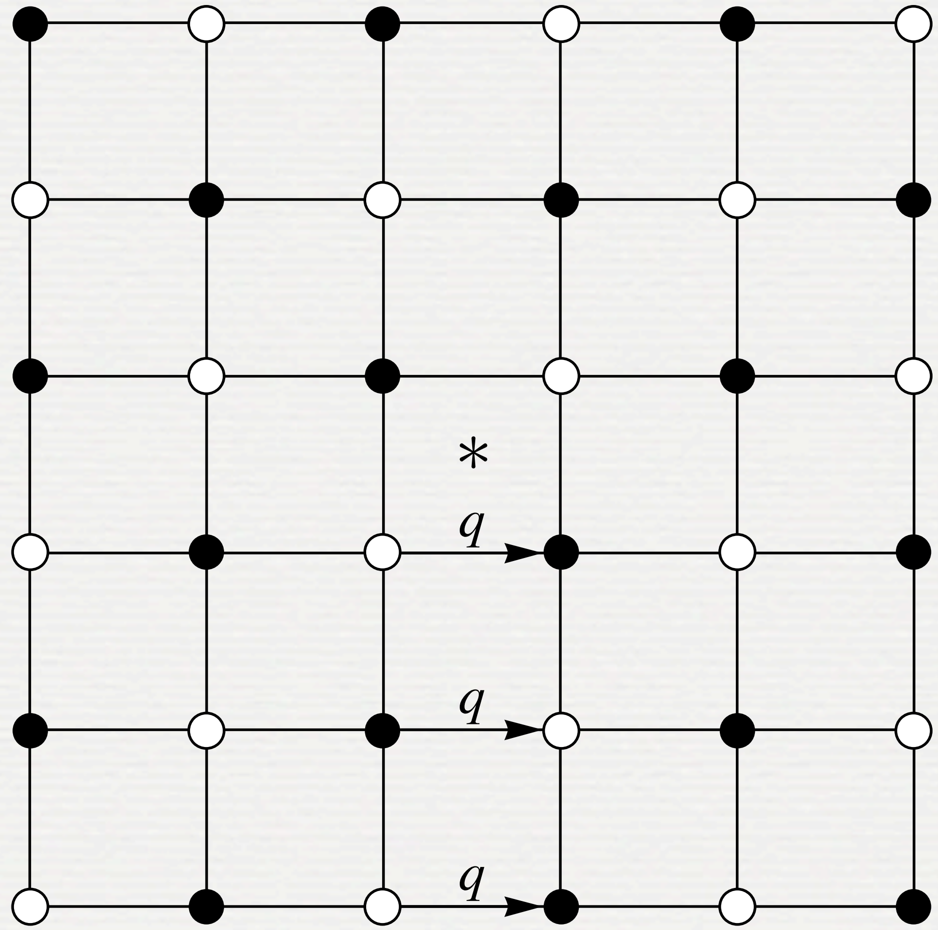


Suppose $qq^* = 1$.

Then $Z(q) = \det \mathbb{K}(q)$ counts loops with weight $q^{\mathrm{Area}} + (q^*)^{\mathrm{Area}}$.

In particular $Z(e^{i\theta})$ counts loops with weight $2\cos(\mathrm{Area}\,\theta)$.

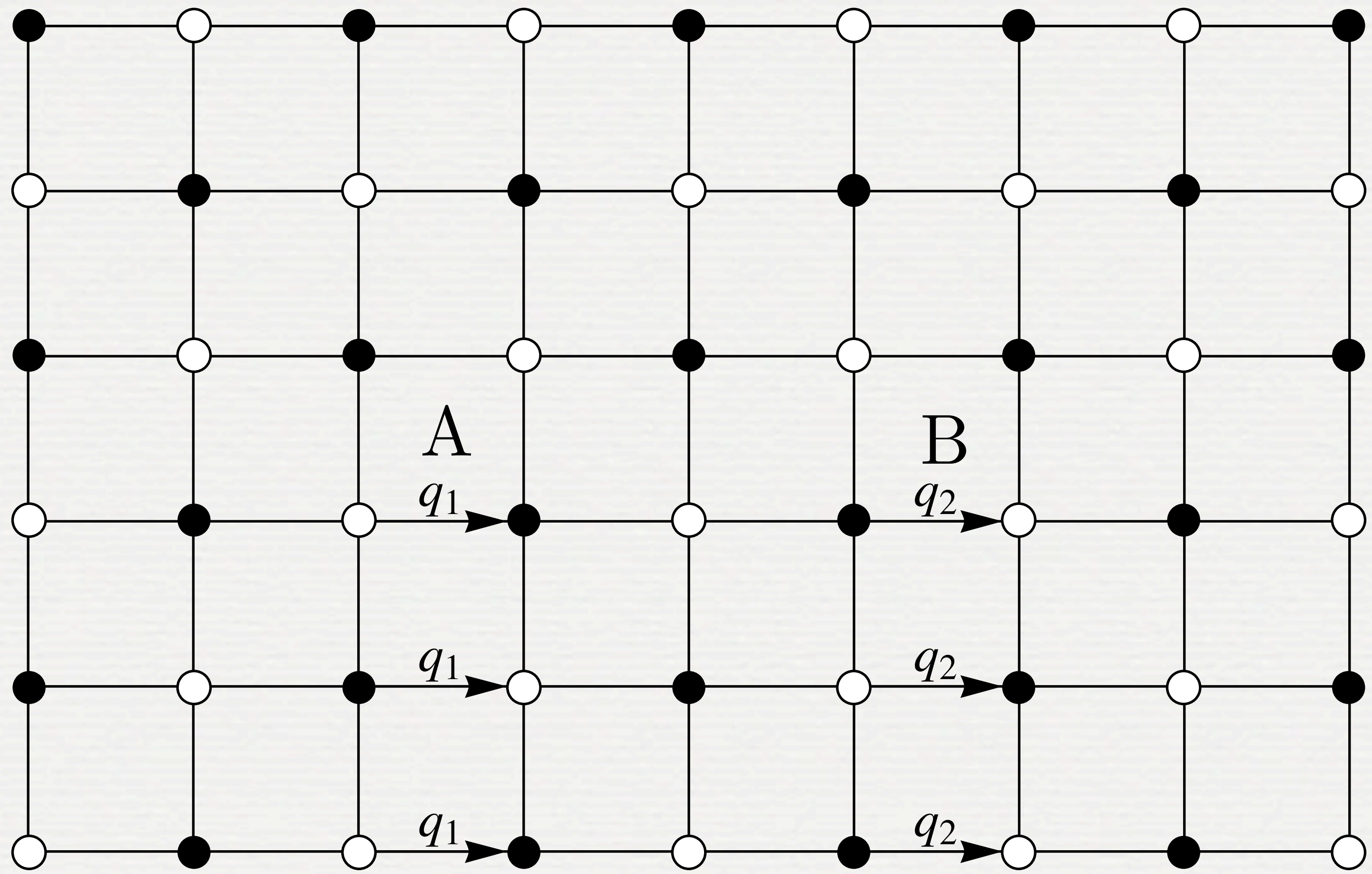
Another example.



Suppose $qq^* = 1$.

Loops surrounding $*$ have weight $q + q^*$.

Another example.



Loops around A: $q_1 + q_1^*$

Loops around B: $q_2 + q_2^*$

Loops around A and B: $q_1 q_2 + (q_1 q_2)^*$

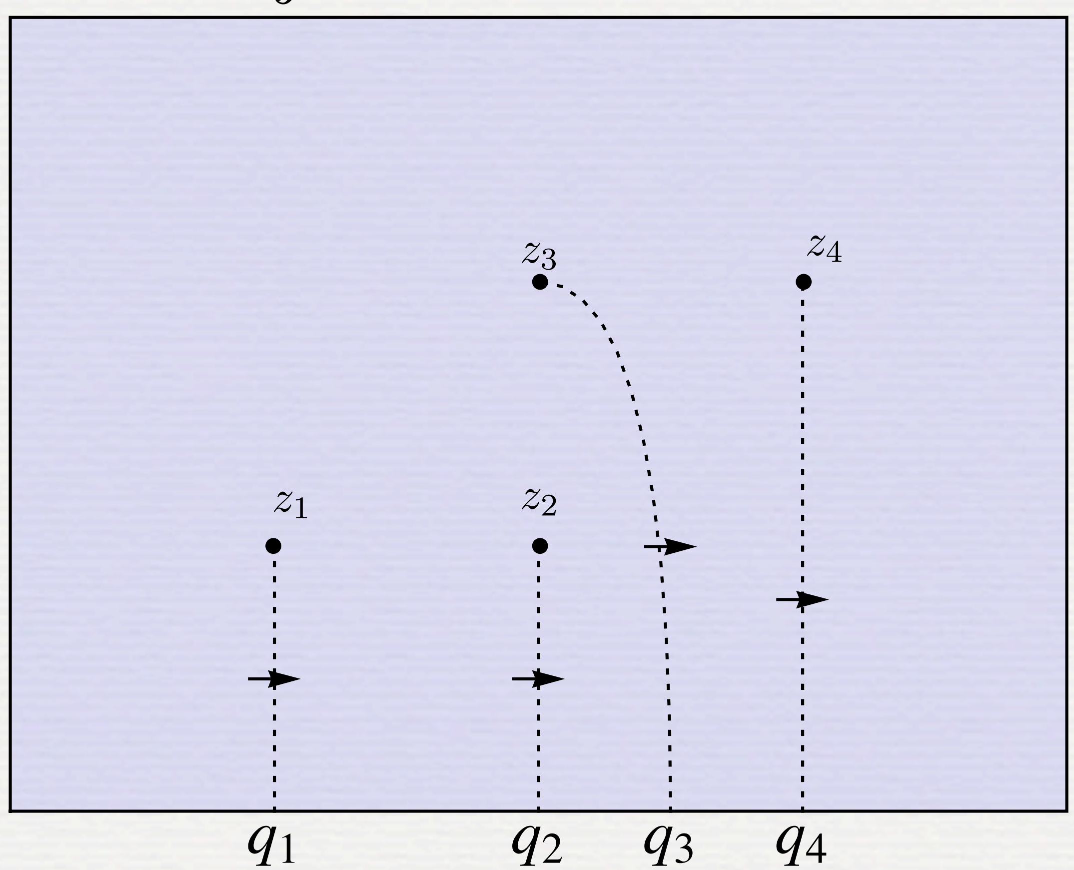
We can choose q_1, q_2 so that these three quantities are algebraically independent.

Lemma (based on [Fock-Goncharov])

By varying the qs one can extract from $\det \mathbb{K}$ the contribution from any finite lamination.

That is, writing $\det \mathbb{K} = \sum_L C_L \prod (m + m^*)$ we have

$$C_L = \int \phi_L \det \mathbb{K} \, dq_1 \dots dq_k.$$



Can one compute $Z(\mathbf{q}) = \det \mathbb{K}(\mathbf{q})$?

Theorem:

$$F(\mathbf{q}) := \lim_{\epsilon \to 0} \frac{Z_\epsilon(\mathbf{q})}{Z_\epsilon(\mathbf{1})} \quad \text{exists and is conformally invariant.}$$

Proof idea:

Take a path of weights $\mathbf{q}_t, 0 \leq t \leq 1$, with $\mathbf{q}_0 = \mathbf{1}$.

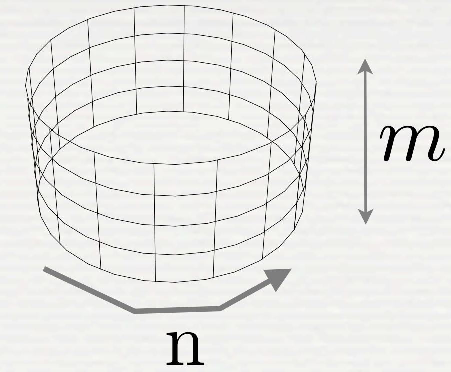
$$\frac{d}{dt} \log Z_\epsilon(\mathbf{q}_t) = \frac{1}{2} \frac{d}{dt} \log \det \tilde{\mathbb{K}}(\mathbf{q}_t)$$

which can be written as a sum along the zippers of

the Green's function $\widetilde{\mathbb{K}}^{-1}(\mathbf{q}_t)$... and $\widetilde{\mathbb{K}}^{-1}(\mathbf{q}_t)$ is a discrete analytic function

(depends analytically on the domain). □

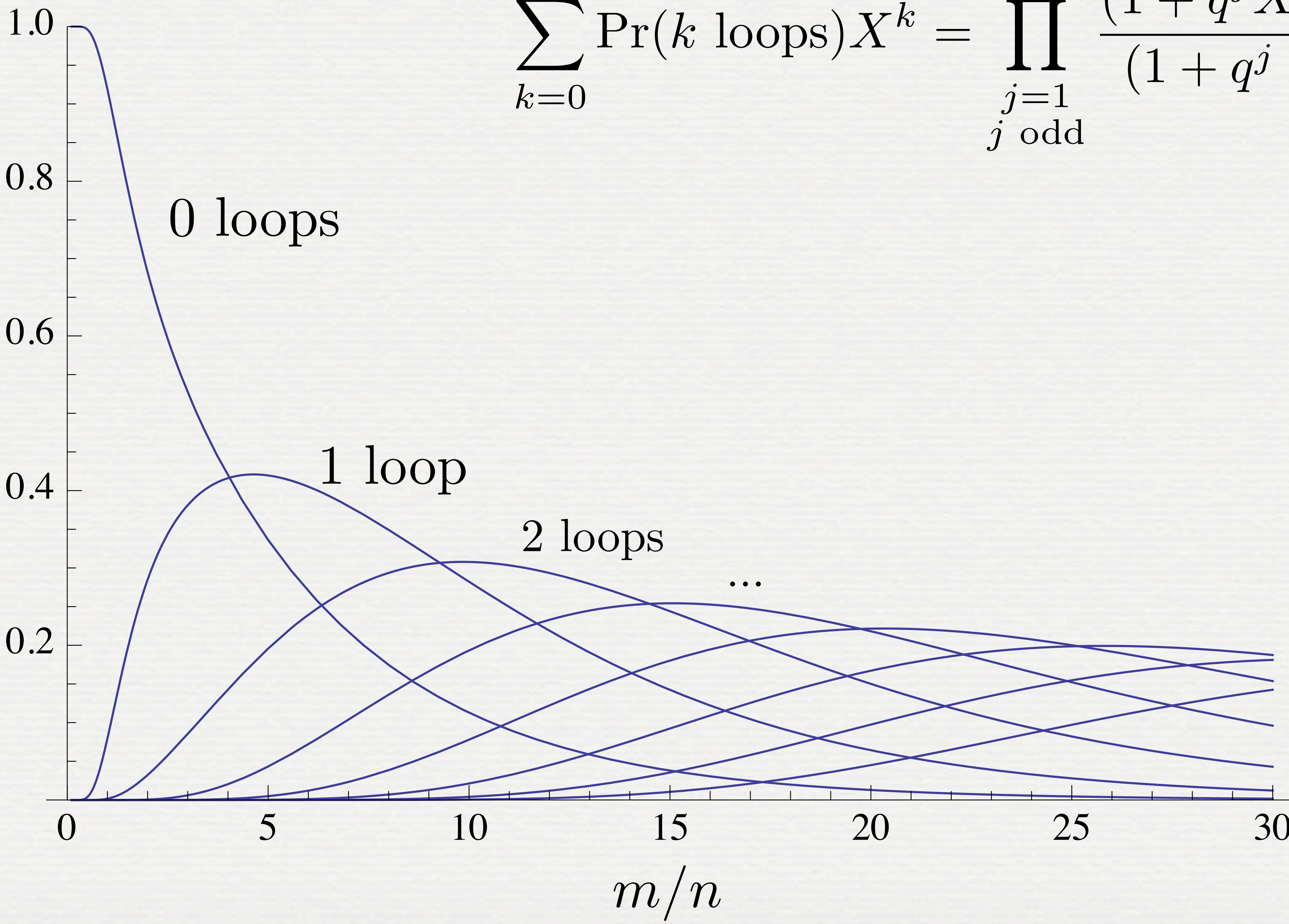
Simple example: $m \times n$ annulus.



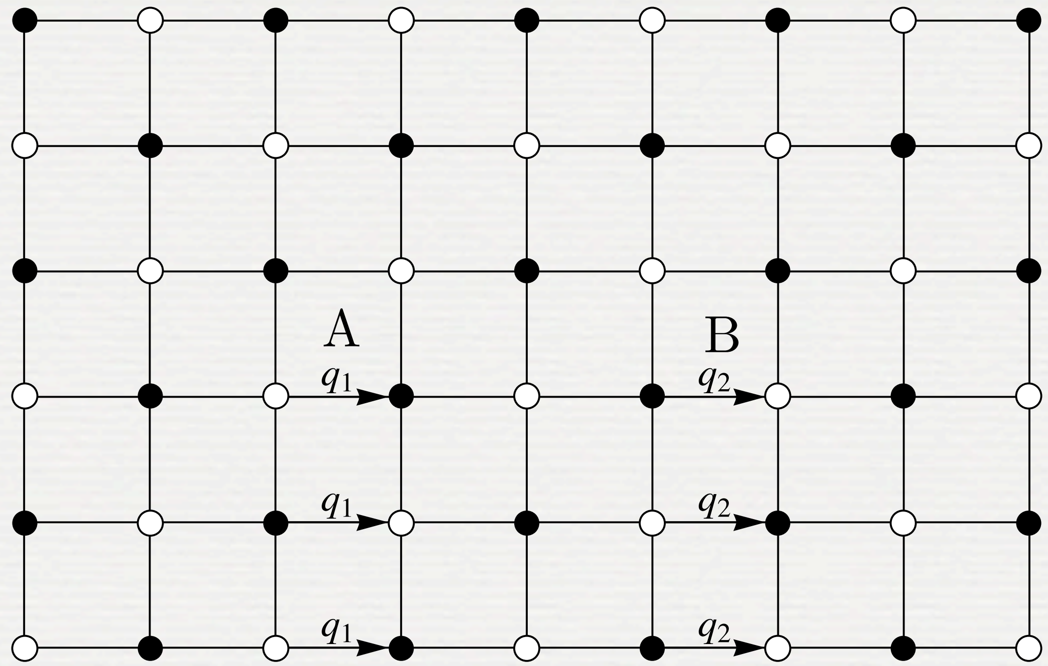
Let $q = e^{-n\pi/m}$.

Then for m even:

$$\sum_{k=0}^{\infty} \Pr(k \text{ loops}) X^k = \prod_{\substack{j=1 \\ j \text{ odd}}}^{\infty} \frac{(1 + q^j X + q^{2j})^2}{(1 + q^j + q^{2j})^2}.$$



Example.



Take $q_1 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $q_2 = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.

Then $Z = \sum_k C_k (2 + t^2)^k$ where k counts the number of loops surrounding both A and B.

Theorem: $\mathbb{E}(k) = g(A, B)$, the Dirichlet Green's function on U.

Extensions.

1. Graphs on curved surfaces?

2. Ising model?

3. Spanning tree/CRSF model. ✓

4. Periodic weights?

The End