Discrete holomorphicity and critical boundary fugacity for the O(*n*) model on the honeycomb lattice

Jan de Gier

University of Melbourne

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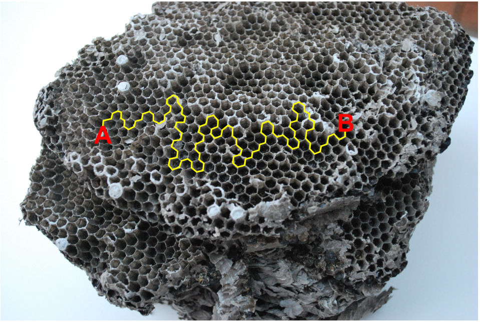
Collaborators:
Nick Beaton
Mireille Bousquet-Mélou
Andrew Elvey Price
Tony Guttmann
Alex Lee

Self avoiding walks



How many ways to walk from A to B without retracing?

Let c_n be the number of SAWs of length n.

Theorem[antiquity]:

$$\log \mu = \lim_{n \to \infty} \frac{1}{n} \log c_n \quad \text{exists}$$

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Theorem[Duminil-Copin and Smirnov 2010]:

$$\mu = \sqrt{2+\sqrt{2}}.$$

Using “discrete parafermions” (and integrability?).

Discrete parafermion

$$F(z) = \sum_{\gamma(a \to z)} \mathrm{e}^{-\mathrm{i}\sigma W(\gamma(a \to z))} x^{\ell} y^{\nu} n^{c}$$

- ℓ: length of the walk
- ν: contacts with the boundary
- n: weight of closed loop ($n = 0$ is SAW)
- W: winding angle
- σ: spin

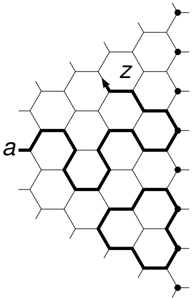

Figure: A configuration γ on a finite domain.

Discrete holomorphicity

Lemma (Smirnov)

For $n \in [-2, 2]$, set $n = 2\cos\theta$ with $\theta \in [0, \pi]$. Then for

$$\sigma = \frac{\pi + 3\theta}{4\pi}, \qquad x^{-1} = 2\cos\left(\frac{\pi - \theta}{4}\right) = \sqrt{2 + \sqrt{2 - n}},$$

the parafermion F with $y = 1$ satisfies the following relation for every vertex v:

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0,$$

where p, q, r are the mid-edges of the three edges adjacent to v.

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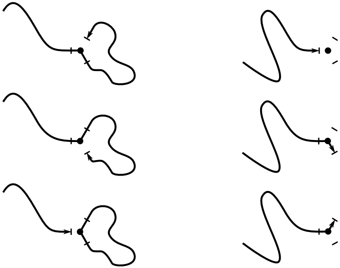
This is a discrete Cauchy integral $\Rightarrow F(z)$ is pre-holomorphic.

Proof of Lemma



The two ways of grouping the configurations which end at mid-edges p, q, r adjacent to vertex v.

- Left: configurations which visit all three mid-edges
- Right: configurations which visit one or two of the mid-edges

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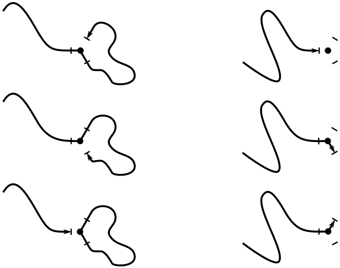


Let

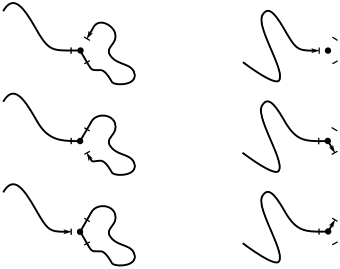
$$\lambda = \mathrm{e}^{-\mathrm{i}\sigma\pi/3}, \qquad j = \mathrm{e}^{2\mathrm{i}\pi/3}.$$

The three contributions on the left add up to zero if

$$-\bar{j}\lambda^4 - j\bar{\lambda}^4 - n = 0.$$

This equation determines the possible values of the parameter σ.

Proof of Lemma



The three contributions on the right add up to zero if

$$-1 - xj\bar{\lambda} - x\bar{j}\lambda = 0.$$

which leads to

$$x^{-1} = 2\cos\left(\frac{\pi}{3}(\sigma - 1)\right).$$

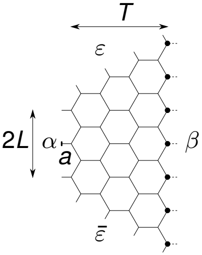
Finite lattice identity

Let us define the following generating functions:

$$A_{T,L}(x,y) = \sum_{\substack{\gamma \subset S_{T,L} \\ a \to \alpha \setminus \{a\}}} x^{\ell} y^{\nu} n^{c},$$

$$B_{T,L}(x,y) = \sum_{\substack{\gamma \subset S_{T,L} \\ a \to \beta}} x^{\ell} y^{\nu} n^{c},$$

$$E_{T,L}(x,y) = \sum_{\substack{\gamma \subset S_{T,L} \\ a \to \varepsilon \cup \bar{\varepsilon}}} x^{\ell} y^{\nu} n^{c},$$



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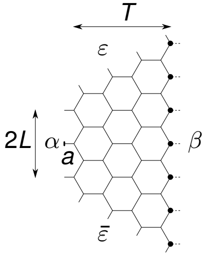

For the special values of $y = 1$ and $x = x_{\text{critical}}$, Smirnov's parafermion implies

$$1 = \cos(\tfrac{3\pi}{8}) A^*_{T,L}(x) + \cos(\tfrac{\pi}{4}) E^*_{T,L}(x) + B^*_{T,L}(x), \qquad A^*_{T,L} = A_{T,L}/Z_{T,L}.$$

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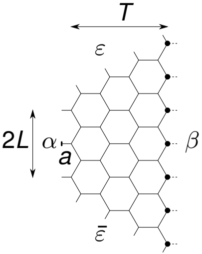

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This important identity provides a bound for $A_T := \lim_{L\to\infty} A_{T,L}$, B_T and E_T.

Sketch of proof $x_{\text{critical}}^{-1} = \sqrt{2+\sqrt{2}}$ for $n = 0$

- For $x < x_c$:

$$B_T(x) < \left(\frac{x}{x_c}\right)^T B_T(x_c) \quad \Rightarrow \quad Z(x) < 2\prod_T (1 + B_T(x))^2 < \infty.$$

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- Consider walks in A_{T+1} that touch the r.h.s. boundary at least once. These are bounded by products of bridges:

$$A_{T+1} - A_T \leq x_c B_T B_{T+1}$$

With $E_T = 0$ and the important identity this implies

$$\cos(\tfrac{3\pi}{8}) x_c B_{T+1}^2 + B_{T+1} \geq B_T \quad \Rightarrow$$

$$B_T(x_c) \geq \frac{const}{T} \quad \Rightarrow \quad Z(x_c) \geq \sum_T B_T(x_c) = \infty.$$

Integrability

The O(n) model on the honeycomb lattice is a solvable lattice model

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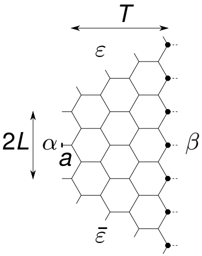
Is there a relationship between integrability and Smirnov's condition?

More examples are known (Cardy, Ikhlef, Fendley).

Boundaries, $y \neq 1$



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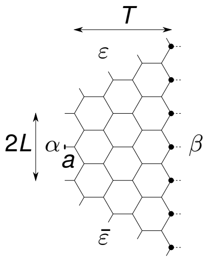


For $n = 2\cos\theta$, Smirnov's parafermion implies

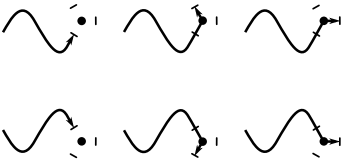
$$1 = \cos\left(\tfrac{3(\pi-\theta)}{4}\right) A^*_{T,L}(x_c, y) + \cos\left(\tfrac{\pi-\theta}{2}\right) E^*_{T,L}(x_c, y) + \frac{y^* - y}{y(y^* - 1)} B^*_{T,L}(x_c, y),$$

with

$$y^* x_c^2 = (2-n)^{-1/2}, \qquad A^*_{T,L} = A_{T,L}/Z_{T,L}.$$

Sketch of proof for $y \neq 1$

At the boundary:



$$\bar{j}\lambda + x_c y j \lambda^2 + x_c y = -(y-1)\bar{j}\lambda,$$
$$j\bar{\lambda} + x_c y \bar{j}\bar{\lambda}^2 + x_c y = -(y-1)j\bar{\lambda}.$$

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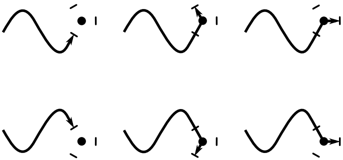

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Summing over all vertices leads to

$$B^*_{T,L}(x_c) \to \left(1 - \frac{y-1}{2yx_c^2}\right) B^*_{T,L}(x_c, y) = \frac{y^* - y}{y(y^*-1)} B^*_{T,L}(x_c, y),$$

Comments

- The value $y = y^*$ is precisely a solution of the Reflection Equation (boundary Yang-Baxter)!

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- For $y = y^*$ the term involving B vanishes:

$$1 = \cos(\tfrac{3(\pi-\theta)}{4})A^*_{T,L}(x_c, y) + \cos(\tfrac{\pi-\theta}{2})E^*_{T,L}(x_c, y) + \frac{y^* - y}{y(y^* - 1)}B^*_{T,L}(x_c, y),$$

Hence B can no longer be bounded by this identity $\Rightarrow$ surface phase transition (adsorption of SAW on the boundary)

Sketch of proof for $n = 0$

In the limit $L \to \infty$:

$$1 = c_\alpha A_T(x_\mathrm{c}, y) + \frac{y^* - y}{y(y^* - 1)} B_T(x_\mathrm{c}, y),$$

implies that for $T \to \infty$

$$B(x_\mathrm{c}, y) = \frac{y(y^* - 1)}{y^* - y}(1 - c_\alpha A(x_\mathrm{c})), \qquad A(x_\mathrm{c}, y) = A(x_\mathrm{c}).$$

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- $1 - c_\alpha A(x_c) > 0$. This implies that

$$B(x_c, y) = \frac{const}{y^* - y}.$$

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- $1 - c_\alpha A(x_c) = 0$. This implies that $B(x_c, y) = 0$ for $y < y^*$. Now use the combinatorial inequality:

$$A_{T+1}(x_c, y) - A_T(x_c, 1) \leq x_c B_T(x_c, 1) B_{T+1}(x_c, y).$$

which implies

$$B_{T+1}(x_c, y^*) \geq \frac{1}{c_\alpha x_c}.$$

Extension off criticality; exponents

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$$\widetilde{F}_\gamma(x) = \mathrm{e}^{\mathrm{i}\tilde{\sigma} W(\gamma)} x^{|\gamma|} n^{c\gamma}.$$

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This can be used to express the winding angle exponent to surface exponents.

Winding angle

Proposition
Let $P(\theta, \ell)$ be the prob. dens. function for winding angles of walks of length ℓ. Then

$$\sum_{\theta} e^{i\tilde{\sigma}\theta} P(\theta, \ell) \sim \mathit{const} \times \ell^{\gamma_{11} - \gamma_1 + 1}.$$

The exponents γ_1 and γ_{11} are defined by

$$\chi_1(x) \sim \mathit{const} \times (1 - x/x_c)^{-\gamma_1},$$
$$\chi_{11}(x) \sim 1 + \mathit{const} \times (1 - x/x_c)^{-\gamma_{11}}.$$

$\chi_1(x)$: walks starting at the surface and ending somewhere in the bulk,
$\chi_{11}(x)$: walks starting and ending at the surface.

Sketch of proof

Define $G_{\theta,\Omega}(x)$ to contain only walks with winding angle θ:

$$G_{\theta,\Omega}(x) = \sum_{\substack{\gamma:a\to\Omega\setminus\partial\Omega \\ W(\gamma)=\theta}} x^{|\gamma|} n^{c(\gamma)}.$$

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Define $H_{\Omega}(x)$ to contain all walks ending on the boundary (these have winding angle associated to the boundary):

$$H_{\Omega}(x) = \sum_{\gamma:a\to\partial\Omega} \mathrm{e}^{\mathrm{i}\tilde{\sigma} W(\gamma)} x^{|\gamma|} n^{c(\gamma)},$$

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The off-critical identity is then written as

$$H^*_{\Omega}(x) + (1 - x/x_c) \sum_{\theta} \mathrm{e}^{\mathrm{i}\tilde{\sigma}\theta} G^*_{\theta,\Omega}(x) = 1.$$

Winding angle exponent

Asume the existence of γ_1 such that

$$\sum_{\theta} G^*_\theta(x) \;\propto\; \chi_1(x) \;\sim\; const \times (1 - x/x_c)^{-\gamma_1}.$$

Assume the existence of γ_{11} such that

$$H^*(x) \;\propto\; \chi_{11}(x) \;\sim\; 1 + const \times (1 - x/x_c)^{-\gamma_{11}}.$$

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The off-critical identity implies now that

$$\sum_\theta \mathrm{e}^{\mathrm{i}\tilde{\sigma}\theta} G^*_\theta(x) \;\propto\; const \times (1 - x/x_c)^{-\gamma_{11}-1}.$$

$$\frac{\sum_\theta \mathrm{e}^{\mathrm{i}\tilde{\sigma}\theta} G^*_\theta(x)}{\sum_\theta G^*_\theta(x)} \;\sim\; const \times (1 - x/x_c)^{-\gamma_{11}+\gamma_1-1}.$$

End of proof.

Conjecture

From Duplantier and Saleur (CFT for winding angle distrib. on the cylinder)

$$\sum_{\theta} \mathrm{e}^{\mathrm{i}\tilde{\sigma}\theta} P(\theta, \ell) \sim \ell^{-\omega},$$

with

$$\omega = \nu\kappa\tilde{\sigma}/2 = \frac{\kappa\tilde{\sigma}^2}{2(4-\kappa)}.$$

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Hence

$$-\gamma_{11} + \gamma_1 - 1 = \omega = \frac{9}{8}\frac{(2-\kappa)^2}{\kappa(4-\kappa)}.$$

This is in agreement with independent predictions (Bray & Moore, Nienhuis, Cardy):

$$\gamma_1 = \frac{\kappa^2 + 12\kappa - 12}{8\kappa(4-\kappa)}, \qquad \gamma_{11} = -\frac{2(3-\kappa)}{\kappa(4-\kappa)}.$$

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We also have results for wedge exponents in a wedge with angle α.

N. Beaton, J. de Gier and A.J. Guttmann,
The critical fugacity for surface adsorption of SAW on the honeycomb lattice is $1+\sqrt{2}$,
accepted in Comm. Math. Phys.; arXiv:1109.0358.

A. Elvey Price, J. de Gier, A.J. Guttmann and A. Lee,
Off-critical parafermions and the winding angle distribution of the O(n) model,
arXiv:1203.2959.