Imaginary Geometry
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MSRI workshop on conformal invariance and statistical mechanics
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Notes taken by Samuel S Watson

We will discuss how SLE curves can be described as flow lines of the Gaussian free field, and will also show how these results can be used to obtain reversibility results for SLE.

Consider a smooth function $h : D \to \mathbb{R}$, where $D \subsetneq \mathbb{C}$. We consider the vector field $e^{ih(x,y)}$, and we define a θ-angle ray of h to be a flow line of $e^{i(h+\theta)}$. Since h is smooth, the rays vary smoothly in the initial angle. Our goal is to study such flow lines for non-smooth h, specifically the GFF.

Recall that the discrete Gaussian free field is a Gaussian random surface model. We obtain a Gaussian measure on functions from some lattice to $\mathbb{R}$, an the covariance is given by Green's functions for simple random walk. The mean height is just the harmonic extension of the given boundary data. In the fine mesh limit, the DGFF converges to a continuum object called the GFF.

See figures from the slides for pictures of flow lines of h, where h is a (linearly interpolated) GFF. These pictures show that the rays depend monotonically on θ (i.e., rays corresponding to larger to θ lie to the left of rays corresponding to smaller θ). Moreover, there is a lot of "empty space" in the picture. In fact, almost all points are not accessible by a flow line, almost surely. As $\kappa \to 4^-$, this picture converges to a single SLE$_4$ curve.

Launching flow lines from differential initial points, we see that flow lines with the same angle merge with one another. Moreover, launching flow lines from two different boundary points and at all different angles gives interesting interactions: each flow line from one point crosses other lines from the other initial point until it merges with its "partner," which has the same angle.

Our paper focuses on flow line interaction, while previous papers address many important questions about the existence and uniqueness of this coupling.

Let x_1 and x_2 be boundary points with $x_1 < x_2$.

Theorem 1. If $\theta_1 > \theta_2$, then $\eta_{\theta_1}^{x_1}$ is to the left of $\eta_{\theta_2}^{x_2}$ almost surely. If $\theta_1 = \theta_2$, they merge. If $\theta_1 < \theta_2$, they cross exactly once.

Moreover, we obtain SLE$_\kappa(\rho)$ by considering the law of one of these curves conditioned on the other.

Theorem 2. SLE$_\kappa(\rho)$ processes are continuous.

Proof. Take three flow lines from the same initial point, and choose the boundary data so that they're all continuous by absolute continuity with respect to regular SLE (Girsanov). Then just notice that the middle process is the SLE$_\kappa(\rho)$ that you want. □

The outer boundary of an $\mathrm{SLE}_{16/\kappa}$ process is described by a certain SLE_κ process for $\kappa \in (0,4)$. This was predicted by Bertrand Duplantier, and is natural for certain values of κ, such as $\kappa = 2$ from relationships to discrete models. In fact a much stronger result is true: the whole $\mathrm{SLE}_{16/\kappa}$ process may be obtained from SLE_κ processes.

Theorem 3. The set of all points accessible by SLE_κ flow lines with angles restricted in $[-\pi/2, \pi/2]$ gives an $\mathrm{SLE}_{16/\kappa}$.

The *fan* is the set of points accessible by flowing at a fixed angle from x with angle in $[-\pi/2, \pi/2]$.

Theorem 4. The fan is a strict subset of the light cone: the probability that the fan contains $\eta'(\tau')$ is zero.

We can generalize this theory to the whole plane, where the resulting flow lines are whole plane $\mathrm{SLE}_\kappa(\rho)$ processes. The flow lines obey interaction rules analogous to those for the flow lines from the boundary.

Space-filling SLE: We can choose a point z in a domain, and launch flow lines of angles $\pm\pi/2$ from z. This divides the domain into two domains. We can then repeat for z in one of these domains, and so on. If we iterate for all z in some dense set, we obtain a space-filling analogue of SLE.

Reversibility: A random curve from a to b in a domain D is said to be reversible if the law of a path from a to b has the same law as a path from b to a (up to time-reparametrization). SLE has long been conjectured to be reversible, since it often arises as a limit of discrete models with natural reversibility.

Reversibility for SLE_κ was shown previously for $\kappa \leq 4$ (Dapeng Zhan).

Theorem 5. SLE_κ is reversible for $\kappa \in (0,4)$.

Idea of proof. Suppose that we have a reverse stopping time τ for η from x to y. Pretend $\eta_[\tau, \infty)$ is a flow linea and compute the conditional law of $\eta[0, \tau]$.

It is $\mathrm{SLE}_\kappa(\kappa - 4; kappa - 4)$ in $D \setminus \eta[\tau, \infty)$ conditioned to exit at $\eta(\tau)$. This implies that $\eta[0, \tau]$ is an SLE_κ from y to $\eta(\tau)$. □

More generally, $\mathrm{SLE}_\kappa(\rho_1, \rho_2)$ processes are reversible for $\rho_1, \rho, 2 \geq \kappa/2 - 4$, which is the threshold for the curve to be boundary filling.

Theorem 6. SLE_κ is reversible for $\kappa \in (4,8)$.

Proof. Left and right boundaries of $\eta \sim \mathsf{SLE}_\kappa$ are $\mathsf{SLE}_{16/\kappa}(\rho_1; \rho_2)$. These we already know how to reverse. Given η_L and η_R, the law of η is that of $\mathrm{SLE}_\kappa(\kappa/2 - 4; \kappa/2 - 4)$. □

So let η be such a process from ∞ to $-\infty$ in $\mathbb{R} \times [0, 1]$, and note that for $z \in \mathbb{R}$, η hits z with probability 1 since η is boundary filling. The key lemma for our proofs states that the law of the future and past of the path before and after hitting z is reflection invariant. To prove this, we reduce to $\mathrm{SLE}_{16/\kappa}(\rho_1;\rho_2)$ reversibility by flow line tricks.

We remark that when $\kappa > 8$, SLE is not time reversible. This can be seen heuristically by looking at the flow line pictures: the SLE closes in around its target point. However, we can fix this:

Theorem 7. $\mathrm{SLE}_\kappa(\kappa/2-4;\kappa/2-4)$ is time reversible, even when $\kappa \geq 8$. Moreover, the time reversal of ordinary SLE_κ is an $\mathrm{SLE}_\kappa(\kappa/2-4;\kappa/2-4)$.

One can summarize the reversibility results with a nice phase-transition diagram (see figure).

Theorem 8. Whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (0,4]$ and $\rho > -2$ have time-reversal symmetry.

This was proved when $\rho = 0$ by Dapeng Zhan. The idea of our proof is to use whole-plane classification of bi-chordal SLE to reduce the result to the chordal case.

Theorem 9. Whole-plane $\mathrm{SLE}_\kappa(\rho)$ have time reversal symmetry when $\kappa \in (4,8)$ and $\rho \geq \kappa/2-4$.

We can prove similar reversibility results for space-filling versions of SLE.

One final observation: The SLE_6 and $\mathrm{SLE}_{8/3}$ fans are reversible, but they are not jointly reversible!