Imaginary Geometry and the Gaussian Free Field

Jason Miller and Scott Sheffield

Microsoft Research and MIT

March 26, 2012

Overview

Part I: Imaginary Geometry and the Gaussian free field

- Interpretation of SLE as a flow line of a random vector field
- Explain how this perspective can be used to study SLE

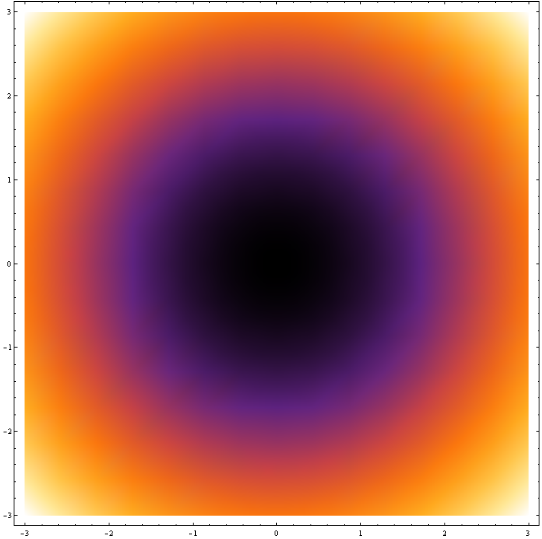
Part II: Reversibility results for SLE

- Chordal SLE
- Space-filling SLE
- Whole-plane SLE

Part I: Imaginary Geometry of the Gaussian Free Field

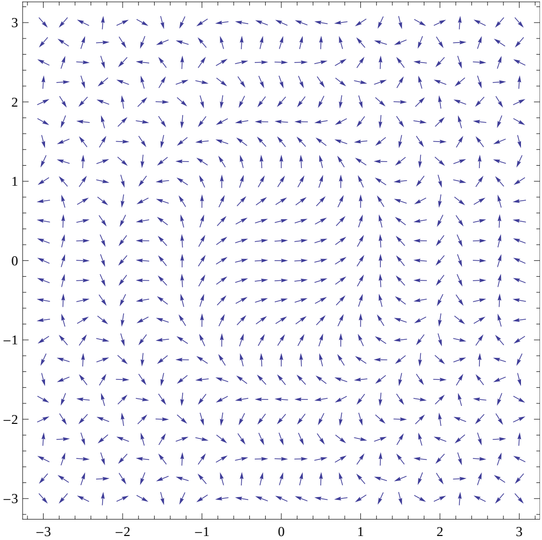
Rays of a Smooth Function

- h smooth $[h(x,y) = x^2 + y^2]$



Rays of a Smooth Function

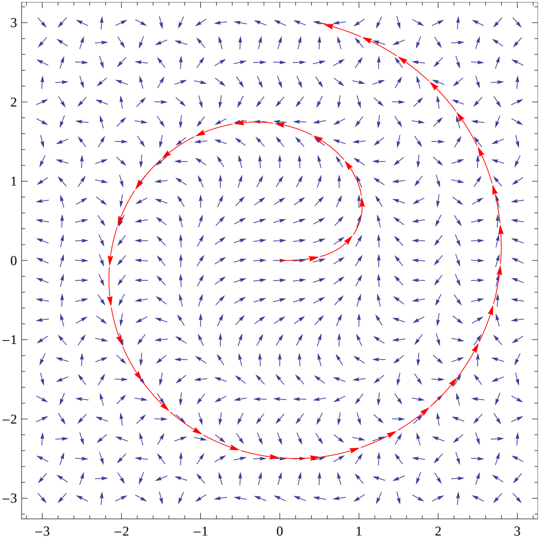
- h smooth $[h(x,y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$



Rays of a Smooth Function

- h smooth $[h(x,y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$
- A **ray** of h is a **flow line** of e^{ih} , i.e. a solution to

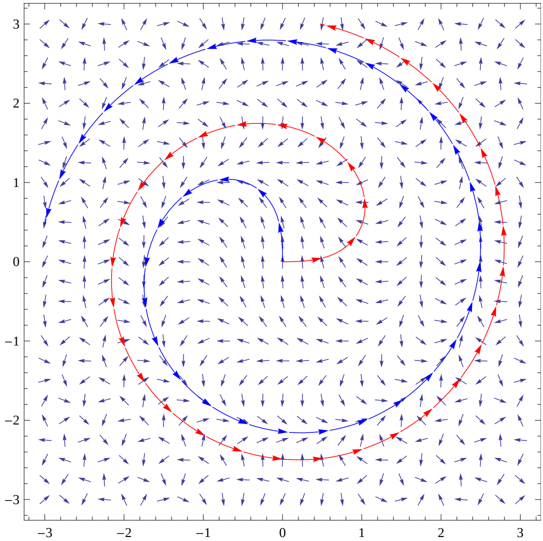
$$\frac{d}{dt}\eta(t) = e^{ih(\eta(t))}$$



Rays of a Smooth Function

- h smooth $[h(x,y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$
- A **θ-angle ray** of h is a **flow line** of $e^{i(h+\theta)}$, i.e. a solution to

$$\frac{d}{dt}\eta(t) = e^{i(h(\eta(t))+\theta)}$$

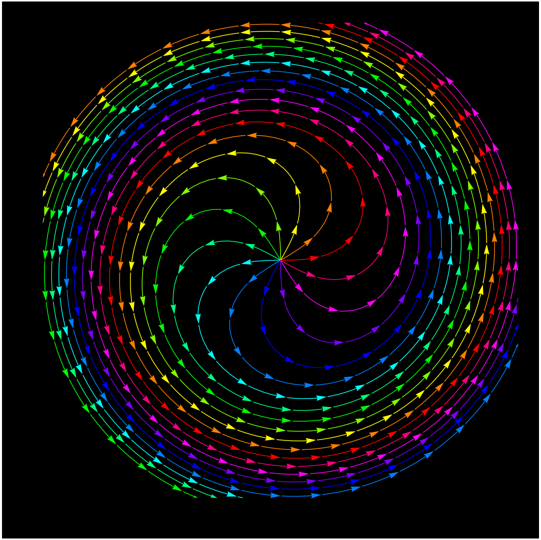


Rays of a Smooth Function

- h smooth $[h(x,y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$
- A **θ-angle ray** of h is a **flow line** of $e^{i(h+\theta)}$, i.e. a solution to

$$\frac{d}{dt}\eta(t) = e^{i(h(\eta(t))+\theta)}$$

- The rays of h vary smoothly and monotonically with θ and are non-intersecting.

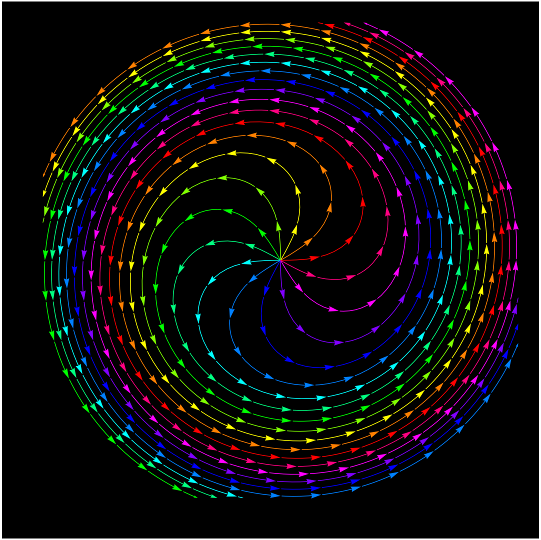


Rays of a Smooth Function

- h smooth $[h(x,y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$
- A **θ-angle ray** of h is a **flow line** of $e^{i(h+\theta)}$, i.e. a solution to

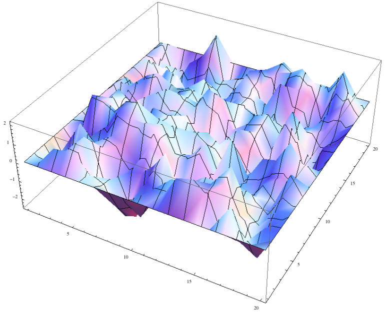
$$\frac{d}{dt}\eta(t) = e^{i(h(\eta(t))+\theta)}$$

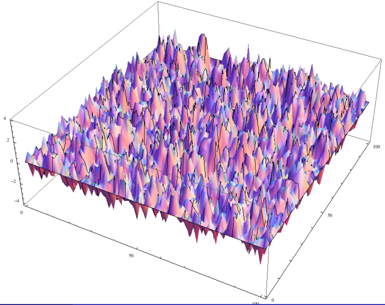
- The rays of h vary smoothly and monotonically with θ and are non-intersecting.
- **Plan:**
  - *Describe the interaction of the rays of the GFF*
  - *Explain how this can be used to study* SLE



The Gaussian Free Field

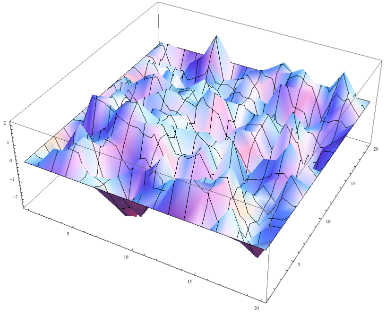
- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.

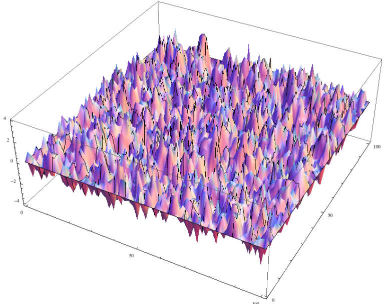




The Gaussian Free Field

- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.
- Gaussian measure on functions $h\colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ where
  - **Covariance**: Green's function for SRW
  - **Mean Height**: harmonic extension of ψ



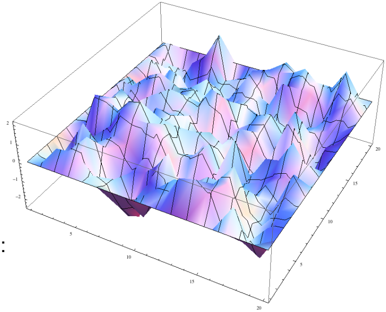


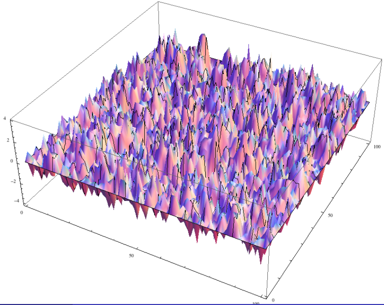
The Gaussian Free Field

- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.
- Gaussian measure on functions $h\colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ where
  - **Covariance**: Green's function for SRW
  - **Mean Height**: harmonic extension of ψ
- Density with respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$

- "Natural perturbation of a harmonic function"





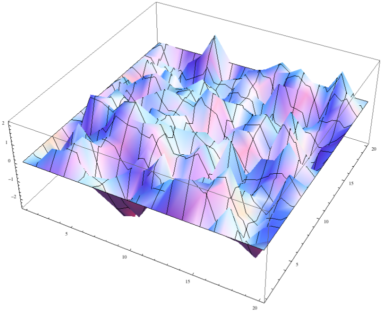
The Gaussian Free Field

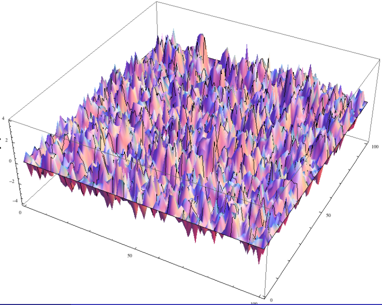
- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.
- Gaussian measure on functions $h\colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ where
  - **Covariance**: Green's function for SRW
  - **Mean Height**: harmonic extension of ψ
- Density with respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$

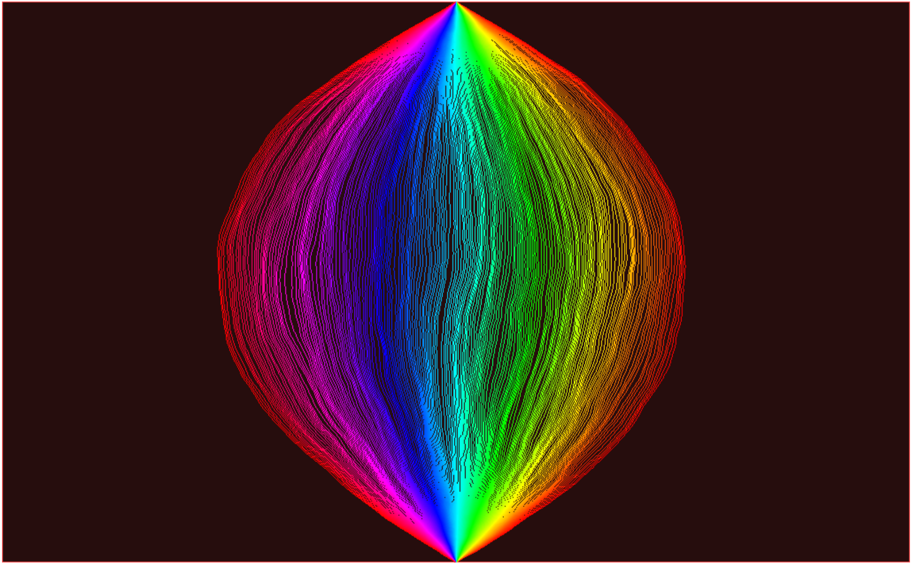
- "Natural perturbation of a harmonic function"
- Fine mesh limit: converges to the continuum GFF, the standard Gaussian with respect to the Dirichlet inner product

$$(f,g)_\nabla = \int \nabla f(x) \cdot \nabla g(x) dx.$$

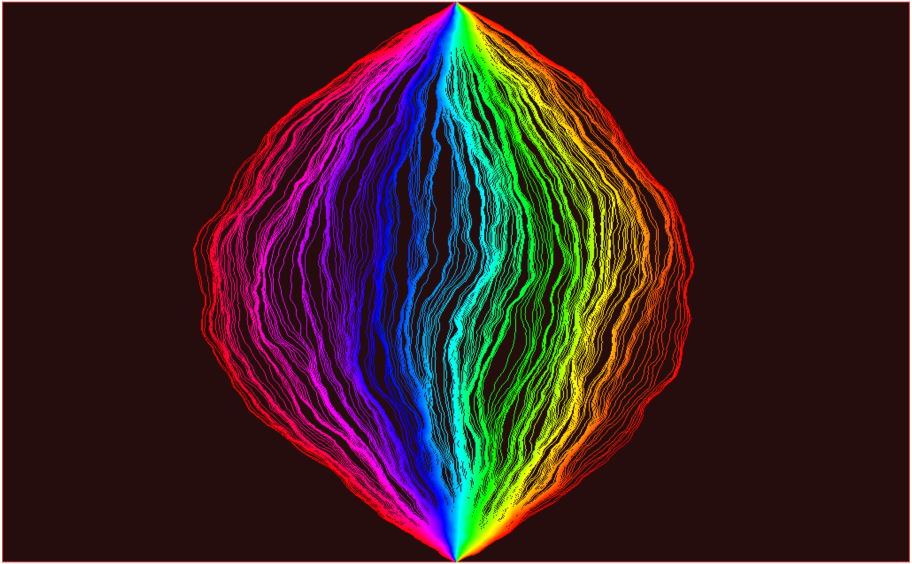




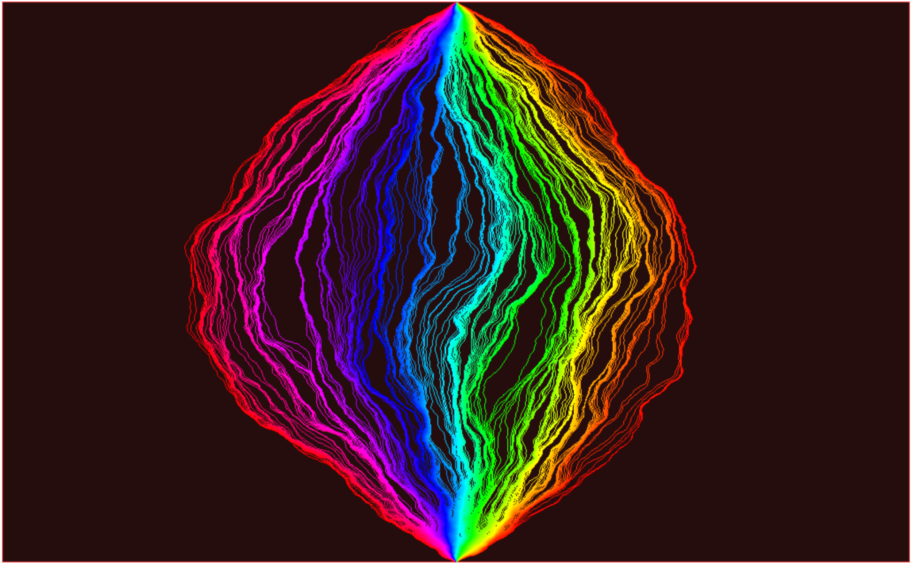
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 31.97$ [$\kappa = 1/256$]



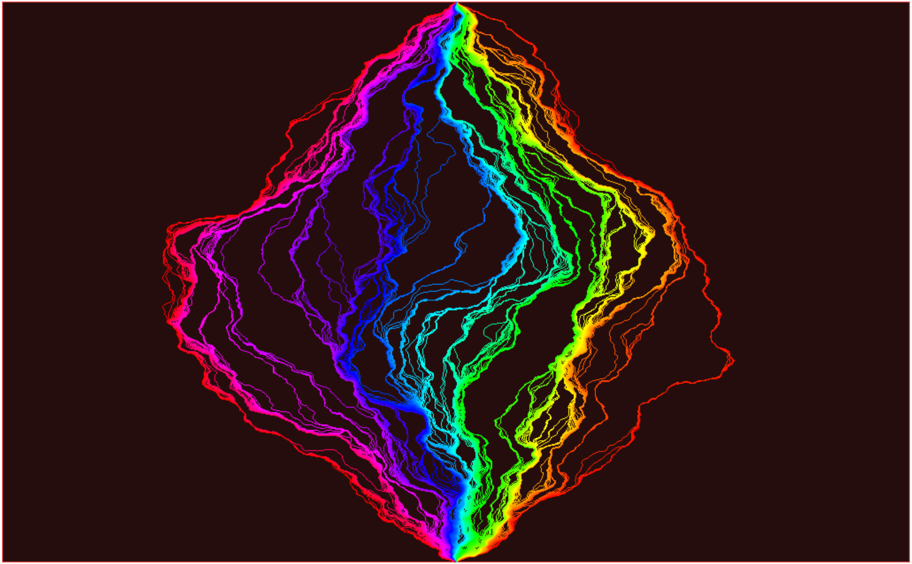
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 11.23$ $[\kappa = 1/32]$



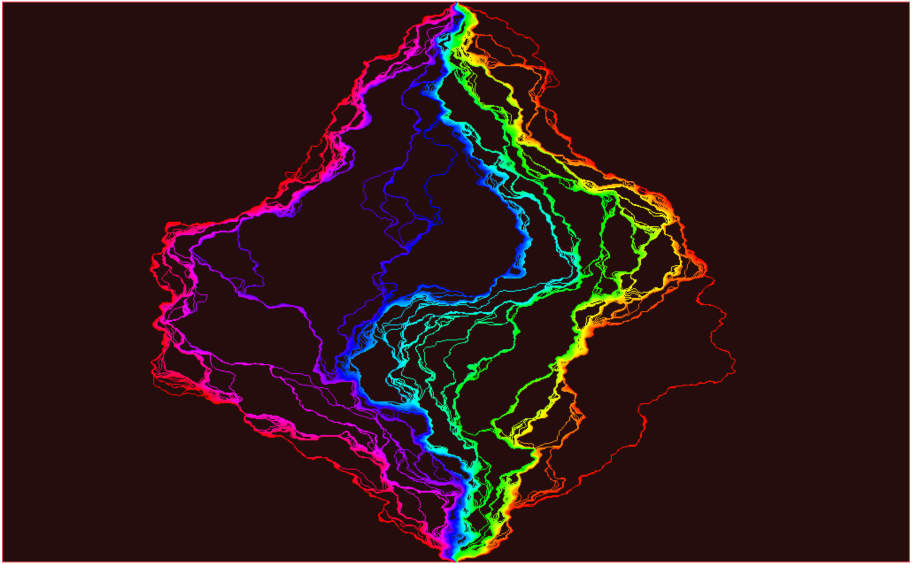
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 7.88$ $[\kappa = 1/16]$



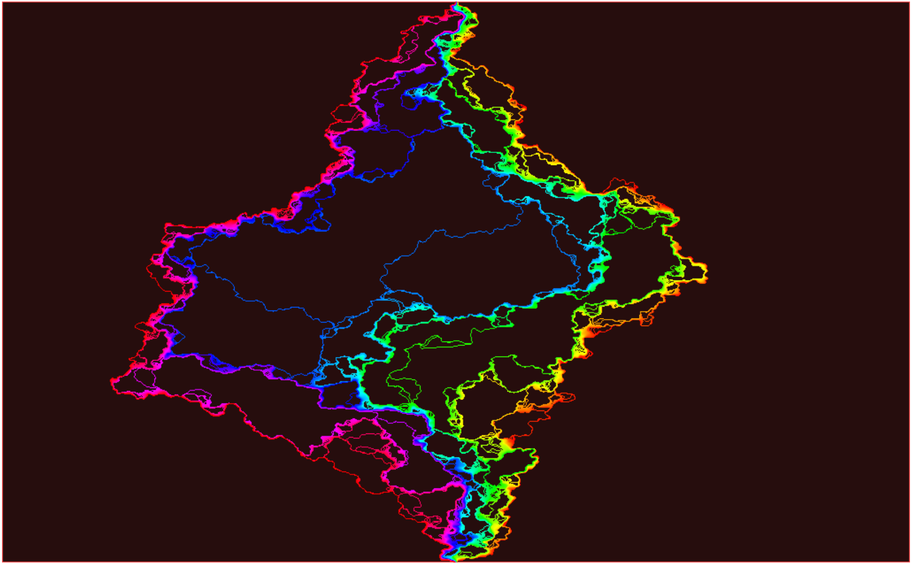
Rays of $e^{ih/\chi}$, h GFF, $\chi = 3.75$ $[\kappa = 1/4]$



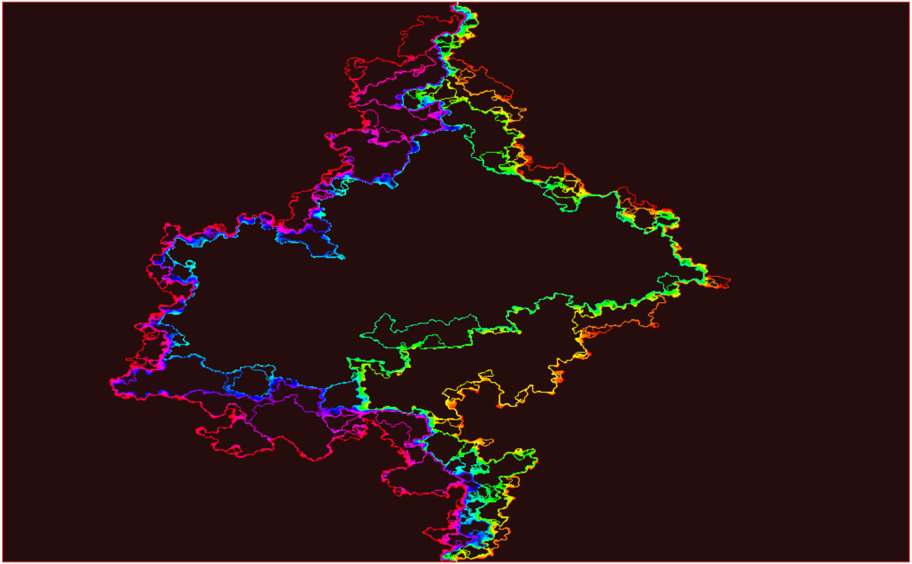
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ [$\kappa = 1/2$]



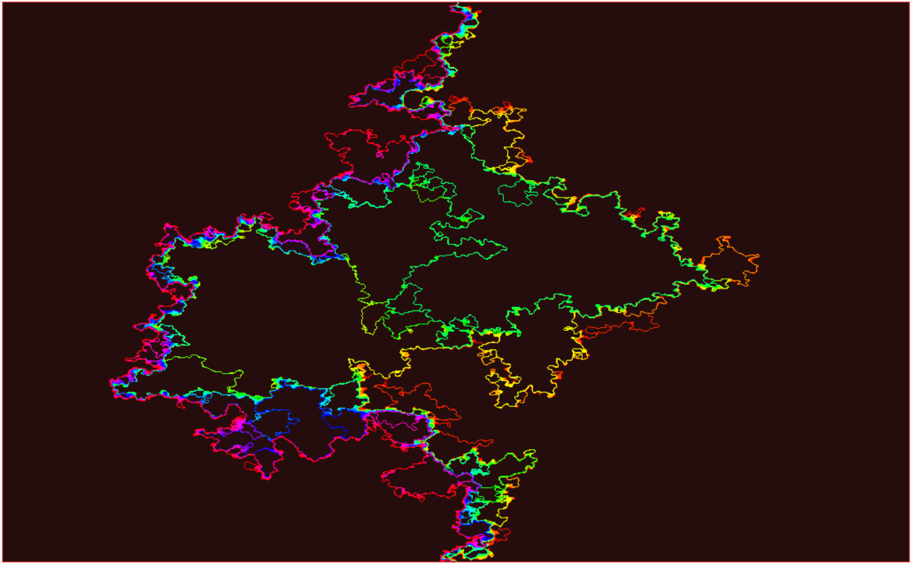
Rays of $e^{ih/\chi}$, h GFF, $\chi = 1.5$ $[\kappa = 1]$



Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 1.02$ [$\kappa = 3/2$]

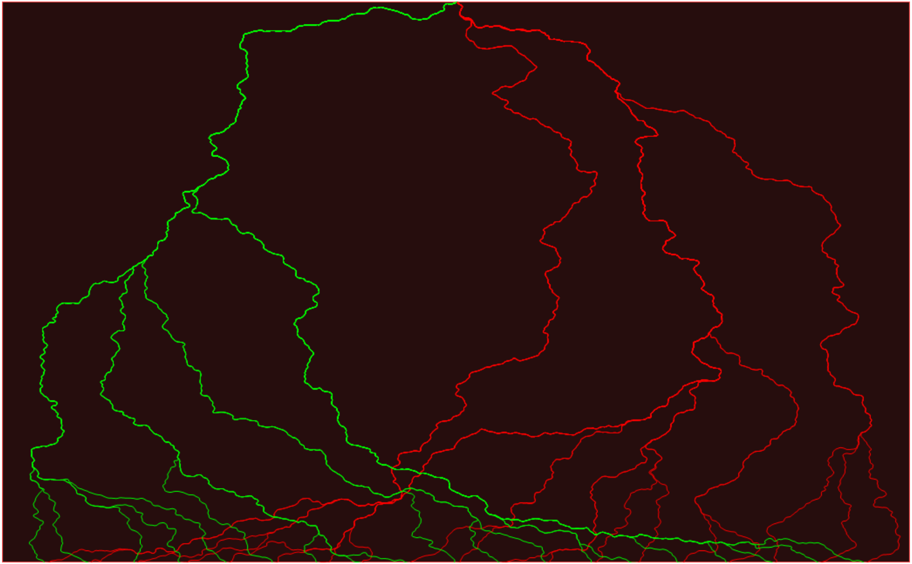


Rays of $e^{ih/\chi}$, h GFF, $\chi = 0.71$ [$\kappa = 2$]

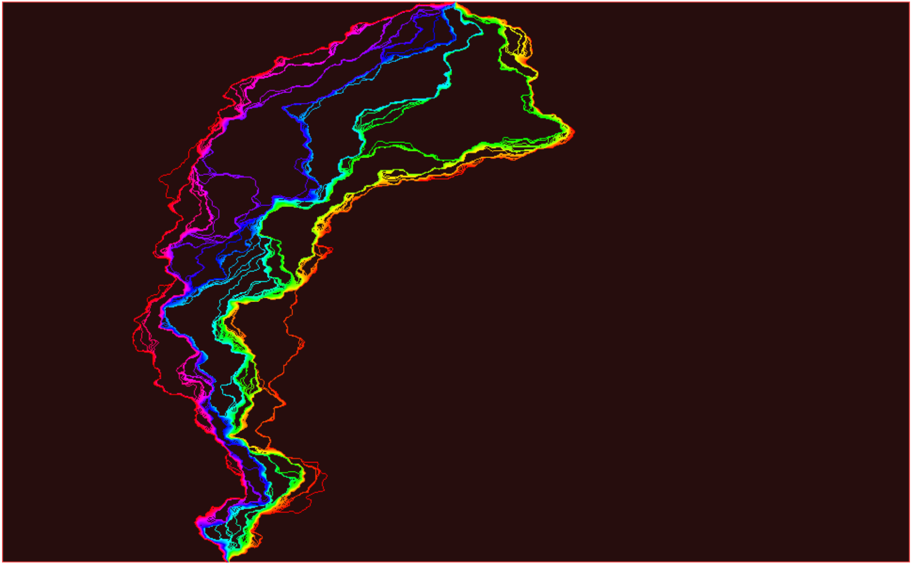


Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ $[\kappa = 1/2]$

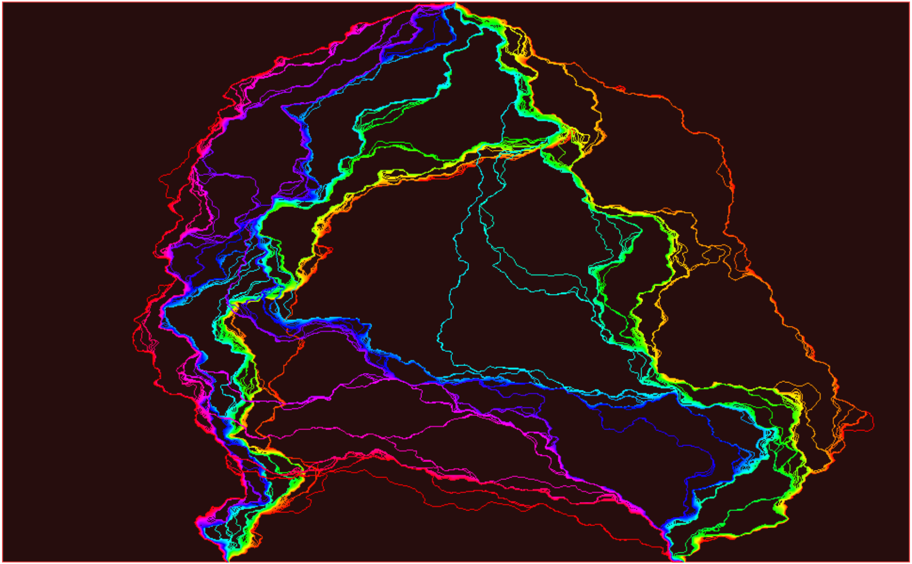
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ [$\kappa = 1/2$]



Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ $[\kappa = 1/2]$



Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ [$\kappa = 1/2$]



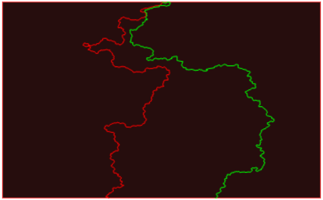
Background

Existence and uniqueness of couplings (η, h) of a GFF h and $\eta \sim \mathrm{SLE}_\kappa$ are studied in the works of Sheffield, Schramm-Sheffield, Dubédat, and Izyurov-Kytölä

New Contributions:

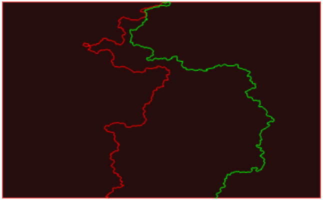
- Developed a robust theory of flow line interaction to make the phenomena observed in the simulations precise
- General forms of SLE duality — the SLE light cone
- $\mathrm{SLE}_\kappa(\underline{\rho})$ processes are continuous (even when they hit the boundary)
  - Important variant of SLE_κ
  - The drift for the driving function includes a linear combination of the Loewner evolution of a collection "force points"
- New reversibility results

Flow line interaction: monotonicity, merging, and crossing



- $\mathbf{x_1}, \mathbf{x_2}$ boundary points with $\mathbf{x_1}$ to the left of $\mathbf{x_2}$
- $\eta_{\theta_1}^{\mathbf{x_1}}$ starting at $\mathbf{x_1}$ with angle θ_1
- $\eta_{\theta_2}^{\mathbf{x_2}}$ starting at $\mathbf{x_2}$ with angle θ_2

Flow line interaction: monotonicity, merging, and crossing



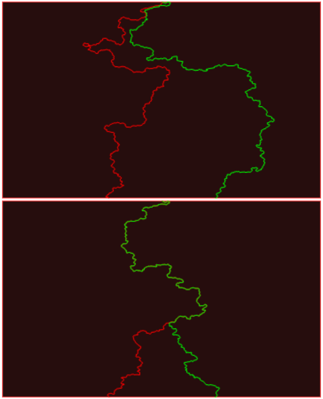
- $\mathbf{x_1}, \mathbf{x_2}$ boundary points with $\mathbf{x_1}$ to the left of $\mathbf{x_2}$
- $\eta_{\theta_1}^{\mathbf{x_1}}$ starting at $\mathbf{x_1}$ with angle θ_1
- $\eta_{\theta_2}^{\mathbf{x_2}}$ starting at $\mathbf{x_2}$ with angle θ_2

Theorem (M., Sheffield)

- $\theta_1 > \theta_2$ *implies* $\eta_{\theta_1}^{\mathbf{x_1}}$ *is to the left of* $\eta_{\theta_2}^{\mathbf{x_2}}$

Flow line interaction: monotonicity, merging, and crossing

- $\mathbf{x_1}, \mathbf{x_2}$ boundary points with $\mathbf{x_1}$ to the left of $\mathbf{x_2}$
- $\eta_{\theta_1}^{\mathbf{x_1}}$ starting at $\mathbf{x_1}$ with angle θ_1
- $\eta_{\theta_2}^{\mathbf{x_2}}$ starting at $\mathbf{x_2}$ with angle θ_2



Theorem (M., Sheffield)

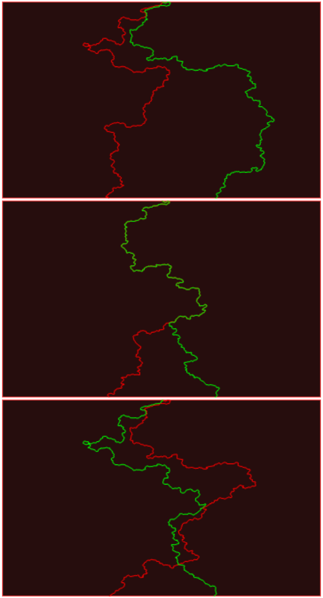
- $\theta_1 > \theta_2$ *implies* $\eta_{\theta_1}^{\mathbf{x_1}}$ *is to the left of* $\eta_{\theta_2}^{\mathbf{x_2}}$
- $\theta_1 = \theta_2$ *implies* $\eta_{\theta_1}^{\mathbf{x_1}}$ *merges with* $\eta_{\theta_2}^{\mathbf{x_2}}$

Flow line interaction: monotonicity, merging, and crossing

- x_1, x_2 boundary points with x_1 to the left of x_2
- $\eta_{\theta_1}^{x_1}$ starting at x_1 with angle θ_1
- $\eta_{\theta_2}^{x_2}$ starting at x_2 with angle θ_2

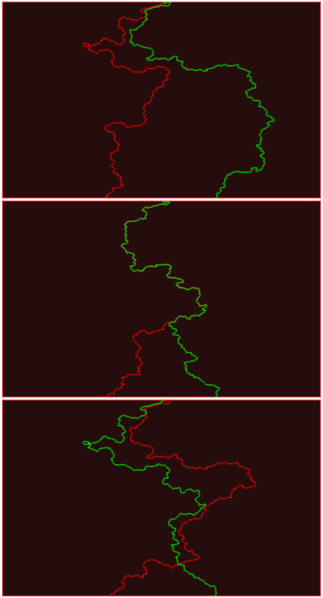
Theorem (M., Sheffield)

- $\theta_1 > \theta_2$ *implies* $\eta_{\theta_1}^{x_1}$ *is to the left of* $\eta_{\theta_2}^{x_2}$
- $\theta_1 = \theta_2$ *implies* $\eta_{\theta_1}^{x_1}$ *merges with* $\eta_{\theta_2}^{x_2}$
- $\theta_1 < \theta_2$ *implies* $\eta_{\theta_1}^{x_1}$ *crosses* $\eta_{\theta_2}^{x_2}$ *exactly once*



Flow line interaction: monotonicity, merging, and crossing

- $\mathbf{x_1}, \mathbf{x_2}$ boundary points with $\mathbf{x_1}$ to the left of $\mathbf{x_2}$
- $\eta_{\theta_1}^{\mathbf{x_1}}$ starting at $\mathbf{x_1}$ with angle θ_1
- $\eta_{\theta_2}^{\mathbf{x_2}}$ starting at $\mathbf{x_2}$ with angle θ_2



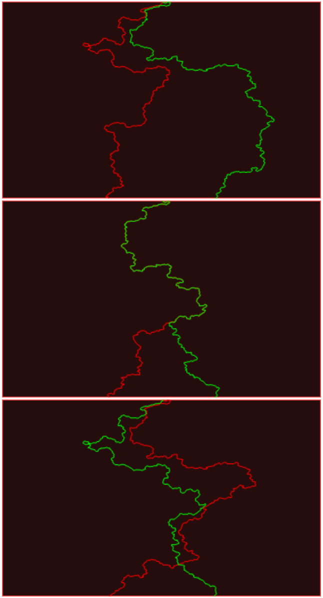
Theorem (M., Sheffield)

- *$\theta_1 > \theta_2$ implies $\eta_{\theta_1}^{\mathbf{x_1}}$ is to the left of $\eta_{\theta_2}^{\mathbf{x_2}}$*
- *$\theta_1 = \theta_2$ implies $\eta_{\theta_1}^{\mathbf{x_1}}$ merges with $\eta_{\theta_2}^{\mathbf{x_2}}$*
- *$\theta_1 < \theta_2$ implies $\eta_{\theta_1}^{\mathbf{x_1}}$ crosses $\eta_{\theta_2}^{\mathbf{x_2}}$ exactly once*

Moreover, $\eta_{\theta_1}^{\mathbf{x_1}}$ given $\eta_{\theta_2}^{\mathbf{x_2}}$ is an $\mathrm{SLE}_\kappa(\underline{\rho})$.

Flow line interaction: monotonicity, merging, and crossing

- x_1, x_2 boundary points with x_1 to the left of x_2
- $\eta_{\theta_1}^{x_1}$ starting at x_1 with angle θ_1
- $\eta_{\theta_2}^{x_2}$ starting at x_2 with angle θ_2



Theorem (M., Sheffield)

- *$\theta_1 > \theta_2$ implies $\eta_{\theta_1}^{x_1}$ is to the left of $\eta_{\theta_2}^{x_2}$*
- *$\theta_1 = \theta_2$ implies $\eta_{\theta_1}^{x_1}$ merges with $\eta_{\theta_2}^{x_2}$*
- *$\theta_1 < \theta_2$ implies $\eta_{\theta_1}^{x_1}$ crosses $\eta_{\theta_2}^{x_2}$ exactly once*

Moreover, $\eta_{\theta_1}^{x_1}$ given $\eta_{\theta_2}^{x_2}$ is an $\mathrm{SLE}_\kappa(\underline{\rho})$.

Generalizes to describe the interaction of any configuration of flow lines with each other and the boundary.

Continuity $\mathrm{SLE}_\kappa(\underline{\rho})$

SLE_κ is continuous ($\kappa \neq 8$: Rohde-Schramm, $\kappa = 8$: Lawler-Schramm-Werner)

Continuity $\mathrm{SLE}_\kappa(\underline{\rho})$

SLE_κ is continuous ($\kappa \neq 8$: Rohde-Schramm, $\kappa = 8$: Lawler-Schramm-Werner)

Theorem (M., Sheffield)

$\mathrm{SLE}_\kappa(\underline{\rho})$ *processes are continuous.*

Continuity $\mathrm{SLE}_\kappa(\underline{\rho})$

SLE_κ is continuous ($\kappa \neq 8$: Rohde-Schramm, $\kappa = 8$: Lawler-Schramm-Werner)

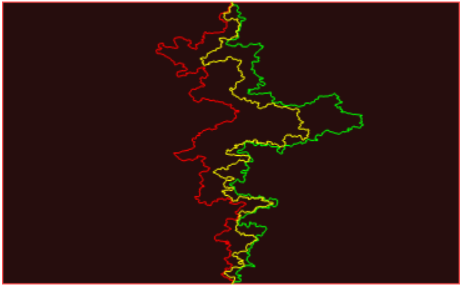
Theorem (M., Sheffield)

$\mathrm{SLE}_\kappa(\underline{\rho})$ *processes are continuous.*

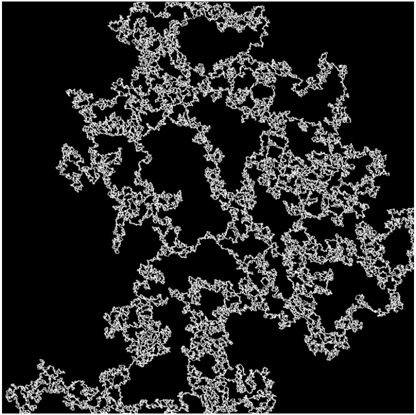
Proof: (for $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ for $\kappa \in (0,4)$) Let η_θ be the flow line with angle θ and fix $\theta_1 > 0 > \theta_2$.

Fact: $\eta_{\theta_1}, \eta_{\theta_2}, \eta_0$ are continuous if they do not hit the boundary.
Reason: Their law is mutually absolutely continuous with respect to SLE_κ.

The law of η_0 given η_{θ_1} and η_{θ_2} is an $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ process. □



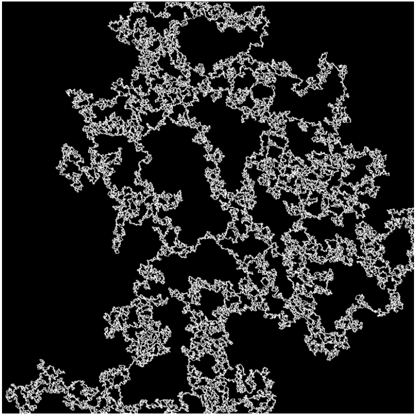
SLE Duality



The outer boundary of an $\mathrm{SLE}_{16/\kappa}$ process is described by a certain SLE_κ process for $\kappa \in (0, 4)$.

- Predicted by Duplantier

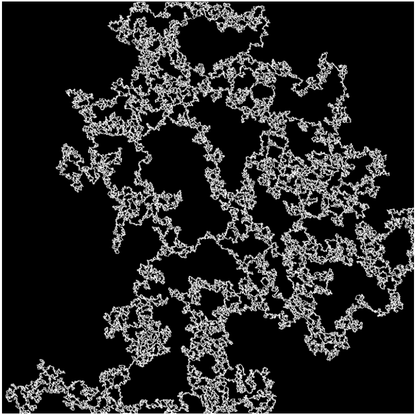
SLE Duality



The outer boundary of an $\mathrm{SLE}_{16/\kappa}$ process is described by a certain SLE_κ process for $\kappa \in (0,4)$.

- Predicted by Duplantier
- Natural for certain values of κ, i.e. $\kappa = 2$ (LERW) and $16/\kappa = 8$ (UST)

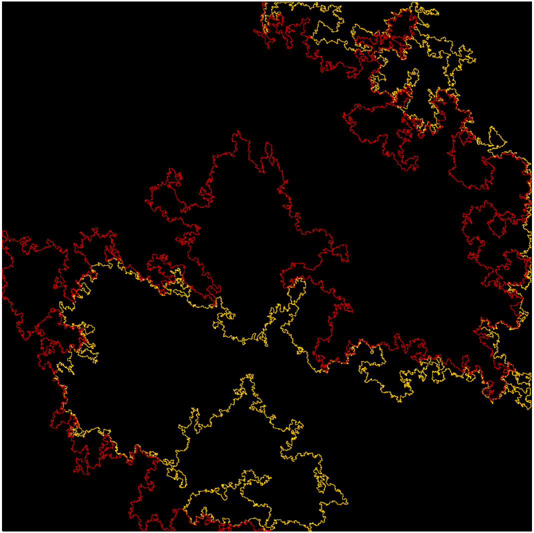
SLE Duality



The outer boundary of an $\mathrm{SLE}_{16/\kappa}$ process is described by a certain SLE_{κ} process for $\kappa \in (0,4)$.

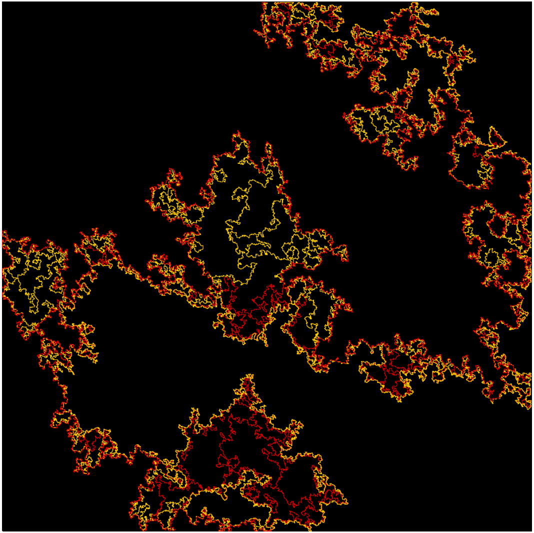
- Predicted by Duplantier
- Natural for certain values of κ, i.e. $\kappa = 2$ (LERW) and $16/\kappa = 8$ (UST)
- Proved in various forms by Zhan and Dubédat

Duality in the Imaginary Geometry: the SLE Light Cone



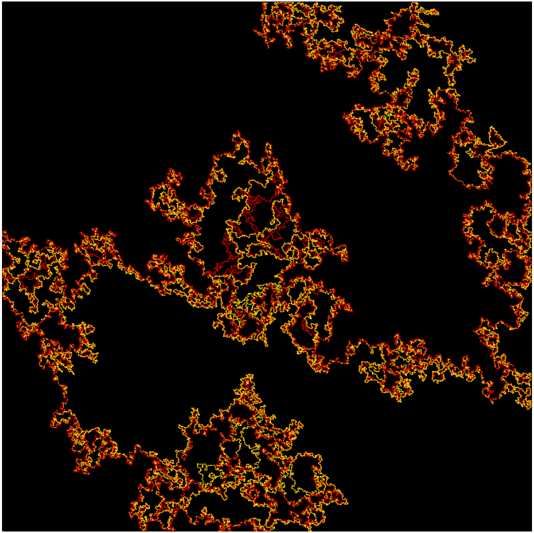
Flow lines with fixed angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

Duality in the Imaginary Geometry: the SLE Light Cone



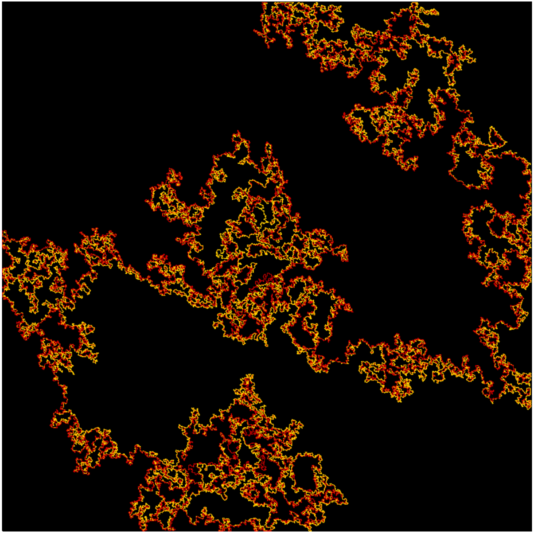
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; one direction change.

Duality in the Imaginary Geometry: the SLE Light Cone



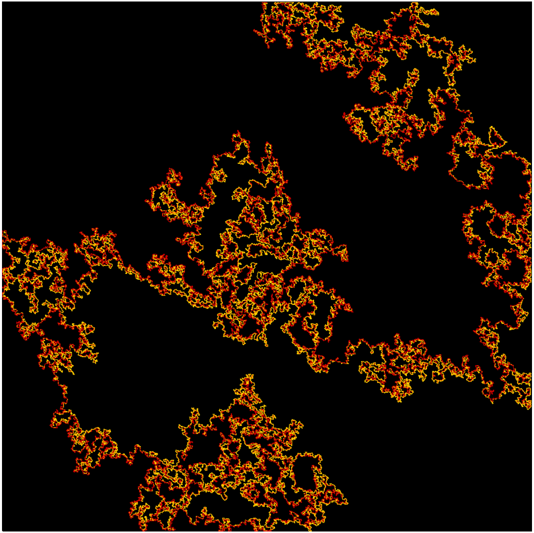
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; two direction changes.

Duality in the Imaginary Geometry: the SLE Light Cone



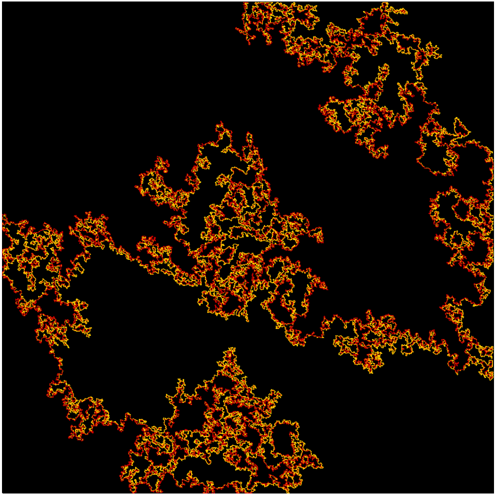
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; three direction changes.

Duality in the Imaginary Geometry: the SLE Light Cone



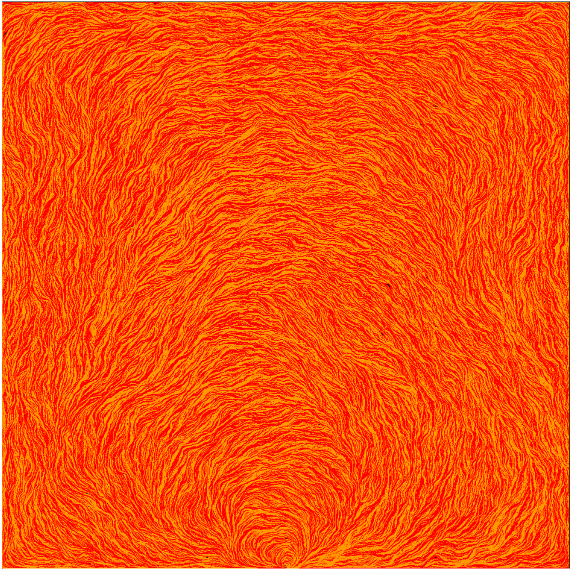
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; four direction changes.

Duality in the Imaginary Geometry: the SLE Light Cone

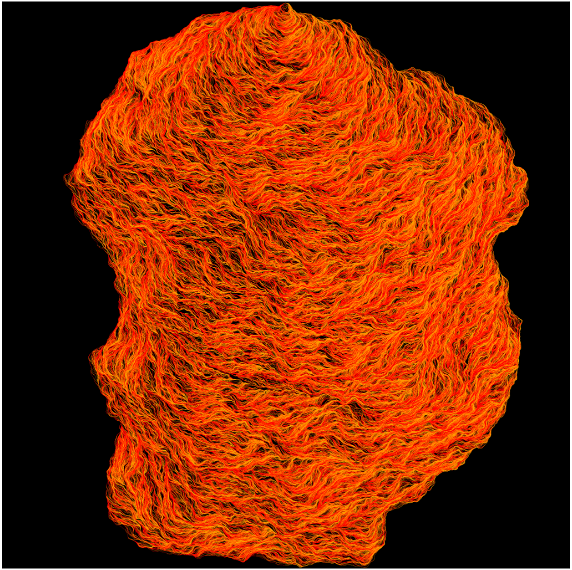


Theorem (M., Sheffield): The set of all point accessible by SLE_κ flow lines ($\kappa \in (0,4)$) with angles restricted in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is an $\mathrm{SLE}_{16/\kappa}$ process.

SLE_{128} Light Cone

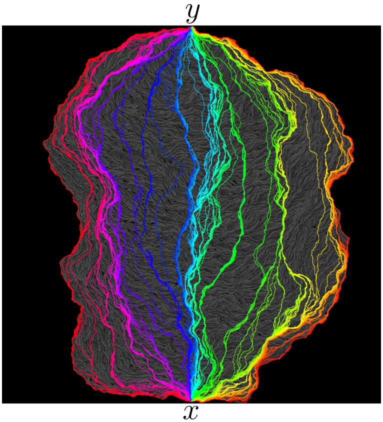


$\mathrm{SLE}_{64}(32;32)$ Light Cone



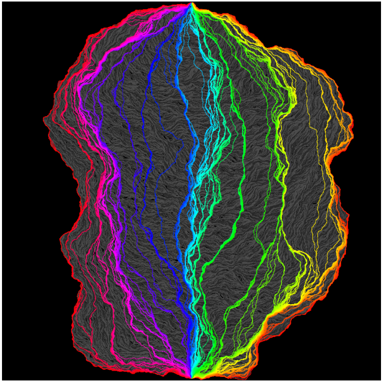
The SLE_κ fan

The fan is the set of points accessible by flowing at a fixed angle from x with angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$; η' an $\mathrm{SLE}_{16/\kappa}$ process from y to x, coupled together using the same GFF.



The SLE_κ fan

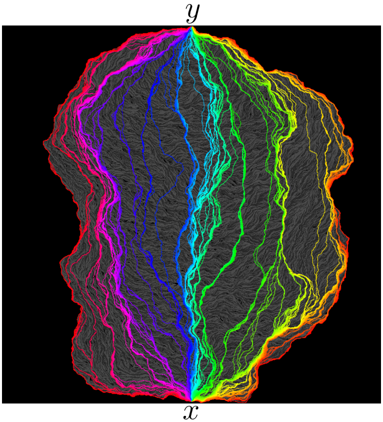
The fan is the set of points accessible by flowing at a fixed angle from x with angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$; η' an $\mathrm{SLE}_{16/\kappa}$ process from y to x, coupled together using the same GFF.



Theorem (M., Sheffield) The fan is a strict subset of the light cone: the probability that the fan contains $\eta'(\tau')$ for any η' stopping time τ' is zero.

The SLE_κ fan

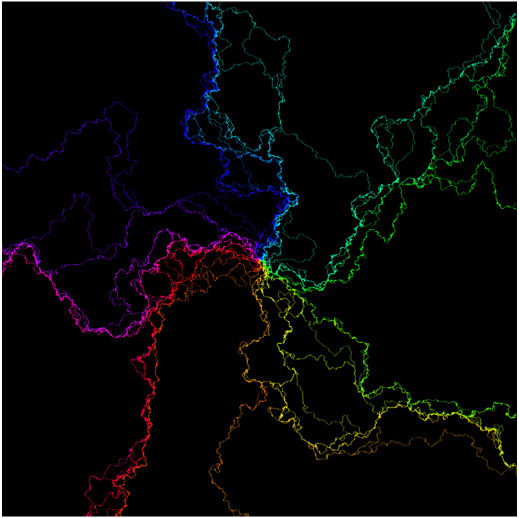
The fan is the set of points accessible by flowing at a fixed angle from x with angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$; η' an $\mathrm{SLE}_{16/\kappa}$ process from y to x, coupled together using the same GFF.



Theorem (M., Sheffield) The fan is a strict subset of the light cone: the probability that the fan contains $\eta'(\tau')$ for any η' stopping time τ' is zero.

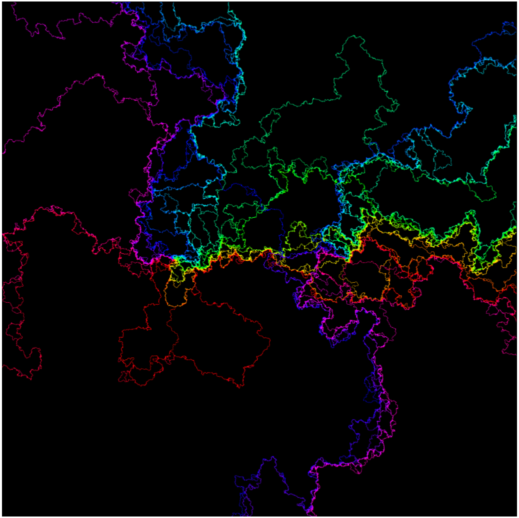
Theorem (M., Sheffield) The fan is a deterministic function of η'

Whole-plane theory



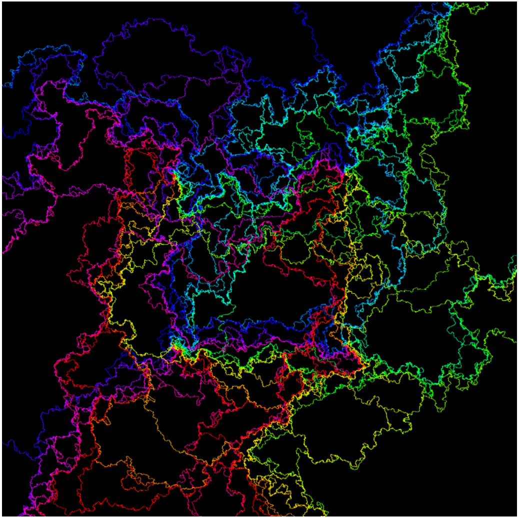
There is a whole-plane version of the theory (M., Sheffield). Flow lines are whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes.

Whole-plane theory



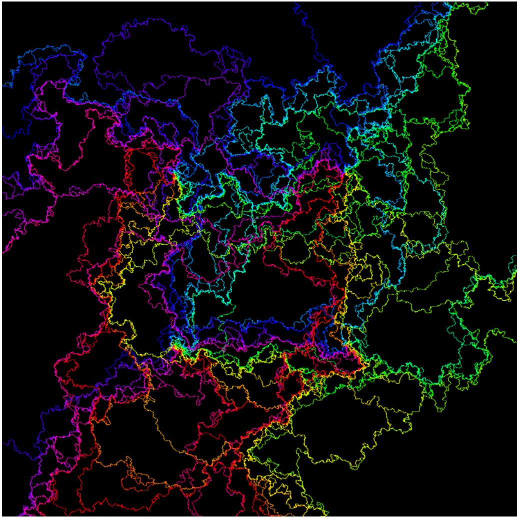
There is a whole-plane version of the theory (M., Sheffield). Flow lines are whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes. Same interaction rules.

Whole-plane theory



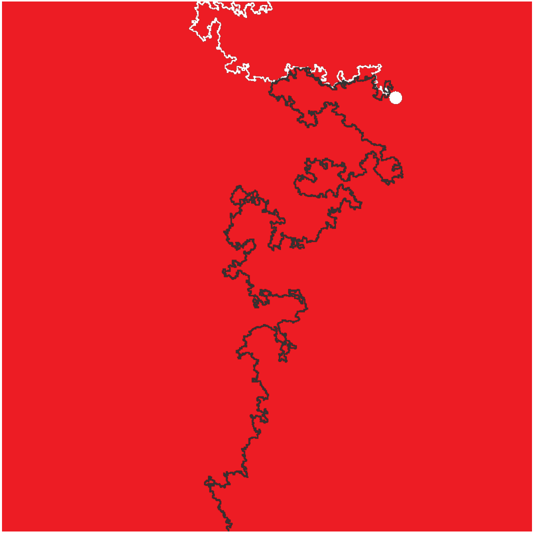
There is a whole-plane version of the theory (M., Sheffield). Flow lines are whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes. Same interaction rules.

Whole-plane theory



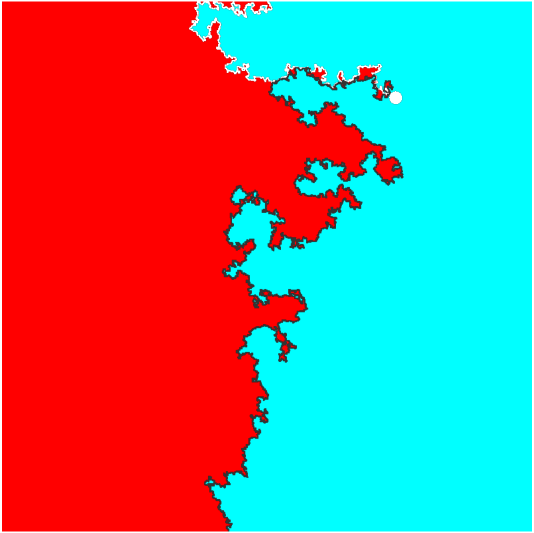
There is a whole-plane version of the theory (M., Sheffield). Flow lines are whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes. Same interaction rules. Versions of the light cone and duality.

Space-filling SLE



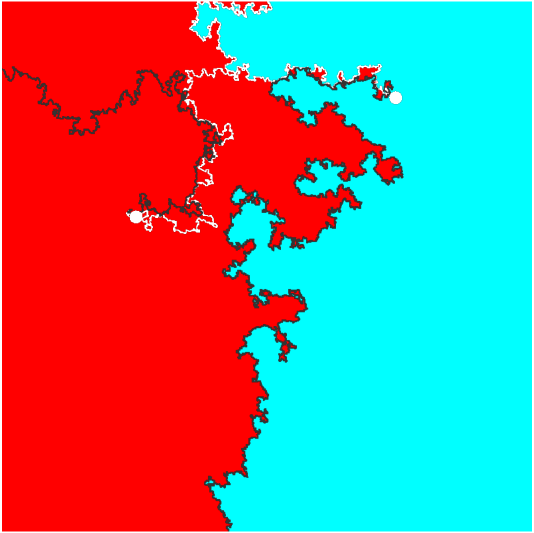
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; one initial point.

Space-filling SLE



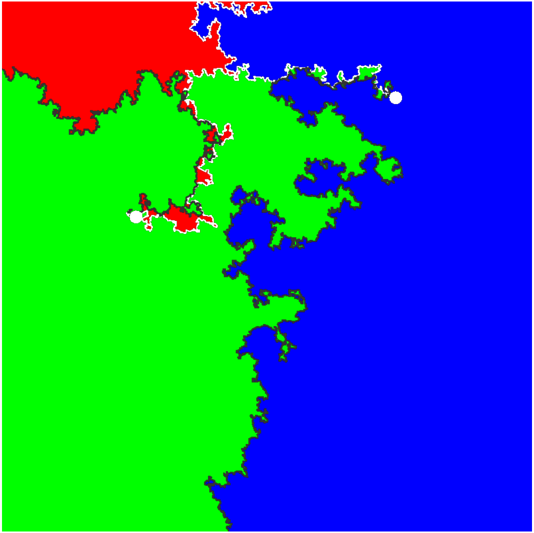
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; one initial point.

Space-filling SLE



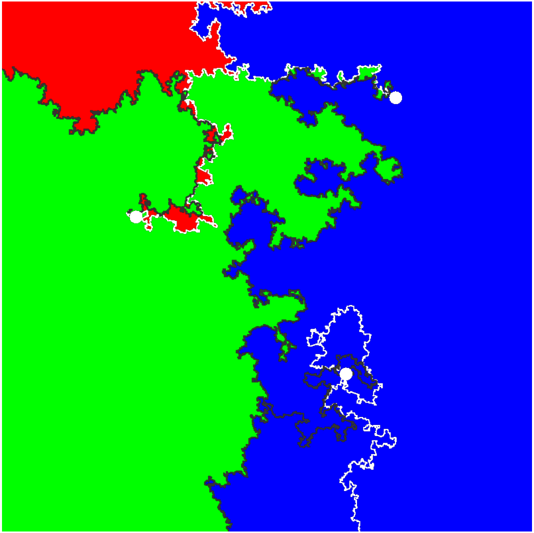
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; two initial points.

Space-filling SLE



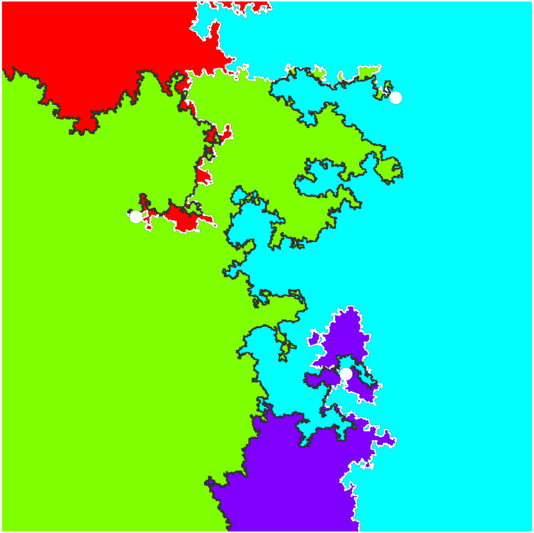
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; two initial points.

Space-filling SLE



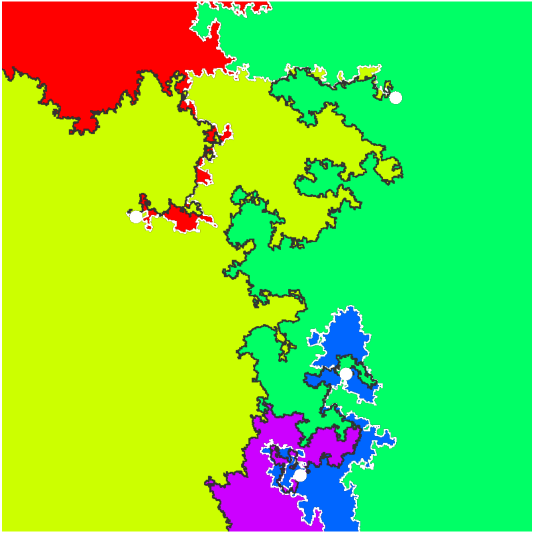
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; three initial points.

Space-filling SLE



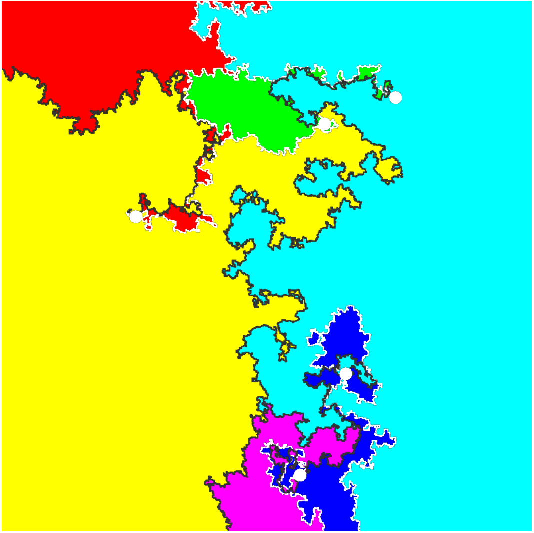
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; three initial points.

Space-filling SLE



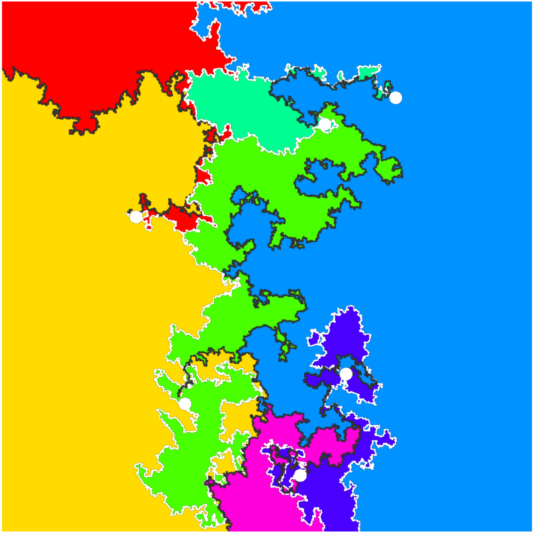
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; four initial points.

Space-filling SLE



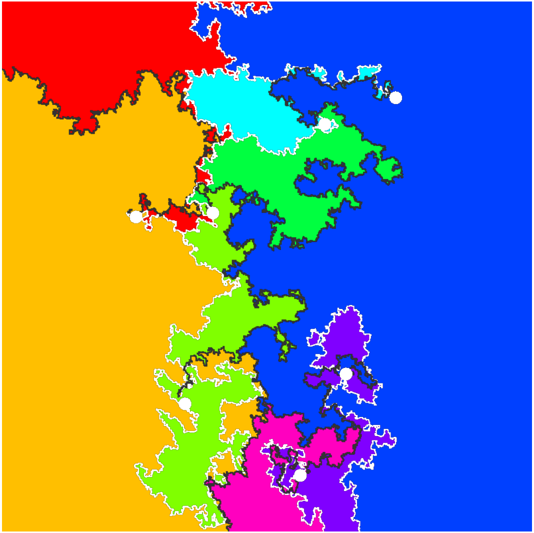
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; five initial points.

Space-filling SLE



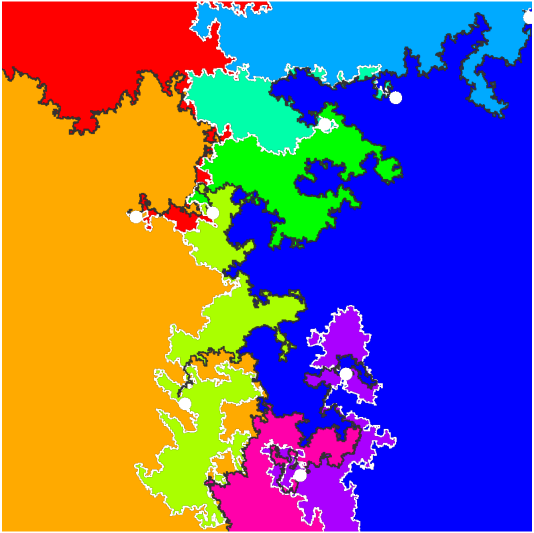
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; six initial points.

Space-filling SLE



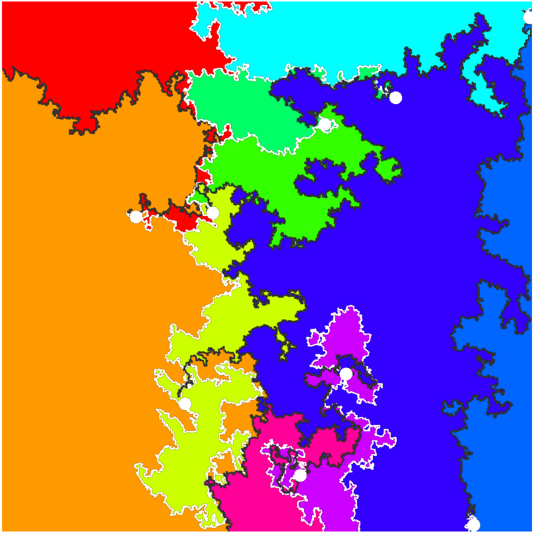
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; seven initial points.

Space-filling SLE



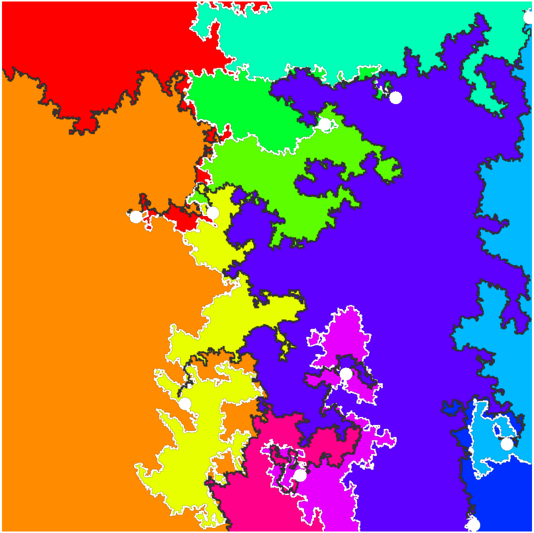
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; eight initial points.

Space-filling SLE



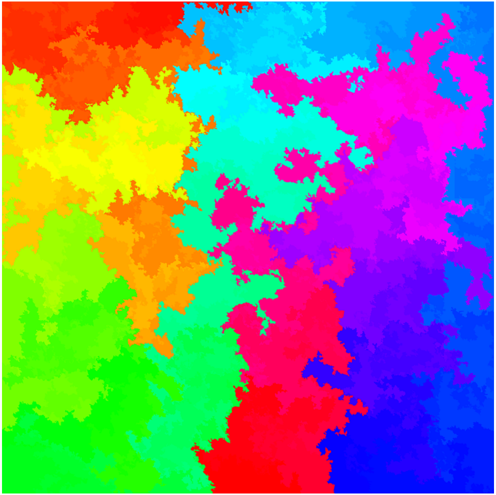
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; nine initial points.

Space-filling SLE



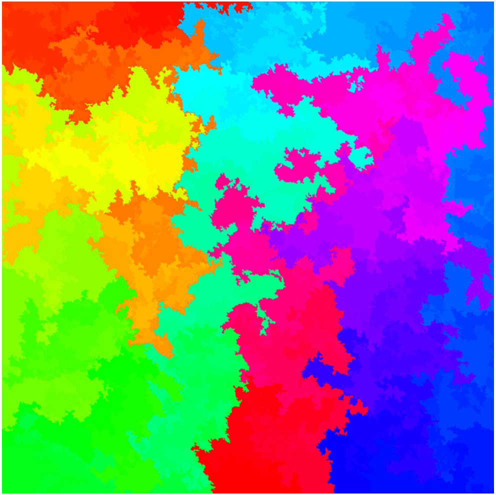
Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; ten initial points.

Space-filling SLE



Three thousand subdivisions.

Space-filling SLE

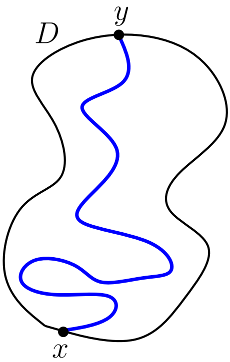


Three thousand subdivisions. **Theorem** (M., Sheffield) This is a continuous curve; a space-filling analog of SLE. It traces the outer boundary of the CLE_κ exploration tree.

Part II: Reversibility

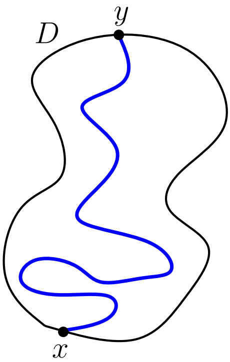
Reversibility

- An SLE_κ η from x to y is said to be **reversible** if the time-reversal of η (parameterized in the reverse direction) has the law of an SLE_κ from y to x.



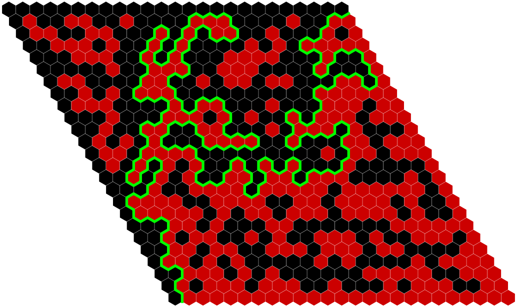
Reversibility

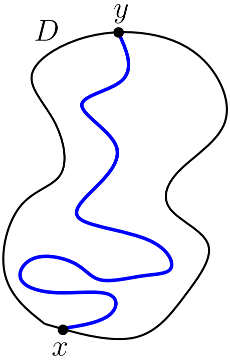
- An SLE_κ η from x to y is said to be **reversible** if the time-reversal of η (parameterized in the reverse direction) has the law of an SLE_κ from y to x.
- Not obvious from the definition of SLE.



Reversibility

- An SLE_κ η from x to y is said to be **reversible** if the time-reversal of η (parameterized in the reverse direction) has the law of an SLE_κ from y to x.
- Not obvious from the definition of SLE.
- Holds for $\kappa = 2, 3, 4, 16/3, 6, 8$ since for these values it is the scaling limit of discrete models with reversibility built in.





Many of the random interfaces which are known or believed to converge to SLE are reversible, in the sense that the reversed path has the same law as the original path (with respect to a slightly modified setup). This motivates the following problem from [76].

Problem 7.3. *Let* γ *be the chordal* SLE_κ *path, where* $\kappa \leq 8$*. Prove that up to reparametrization, the image of* γ *under inversion in the unit circle (that is, the map* $z \mapsto 1/\bar{z}$*) has the same law as* γ *itself.*

Oded Schramm, 2006 ICM proceedings

Reversibility for $\kappa \in (0,4]$

Theorem (Zhan)

SLE_κ *is reversible for* $\kappa \in (0,4]$.

Reversibility for $\kappa \in (0, 4]$

Theorem (Zhan)

SLE_κ *is reversible for* $\kappa \in (0, 4]$.

Theorem (Dubédat, Zhan)

Non-boundary intersecting $\mathrm{SLE}_\kappa(\rho)$ *is reversible for* $\kappa \in (0, 4]$.

Reversibility for $\kappa \in (0, 4]$

Theorem (Zhan)

SLE_κ *is reversible for* $\kappa \in (0, 4]$.

Theorem (Dubédat, Zhan)

Non-boundary intersecting $\mathrm{SLE}_\kappa(\rho)$ *is reversible for* $\kappa \in (0, 4]$.

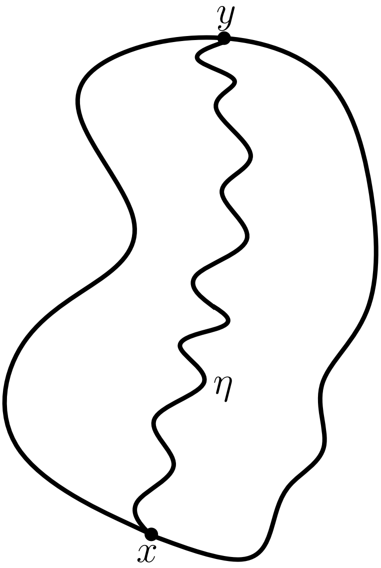
Theorem (M., Sheffield)

$\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ *processes are reversible for* $\kappa \in (0, 4]$, *even when they intersect the boundary.*

- Based on imaginary geometry techniques
- New proof for SLE_κ, $\kappa \in (0, 4]$
- Description of the time reversal of $\mathrm{SLE}_\kappa(\underline{\rho})$ processes

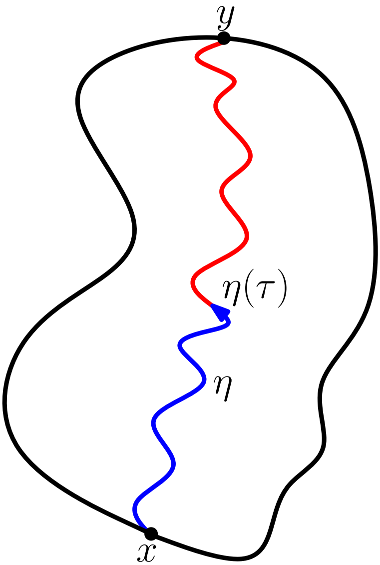
Proof of the reversibility of SLE_κ for $\kappa \in (0,4]$

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D



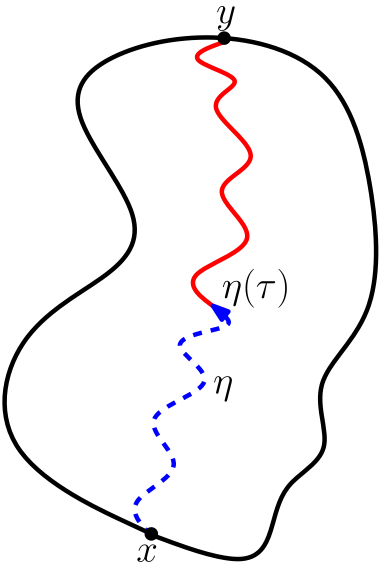
Proof of the reversibility of SLE_κ for $\kappa \in (0,4]$

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D
- τ a **reverse** stopping time for η



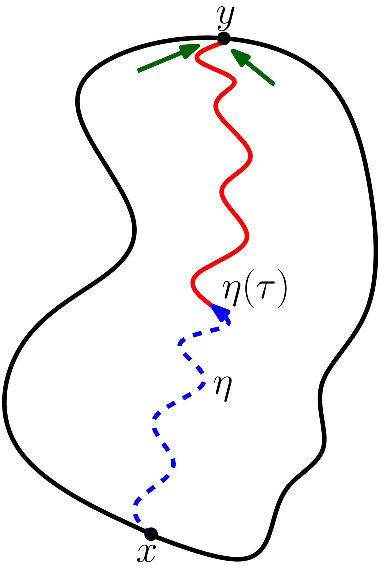
Proof of the reversibility of SLE_κ for $\kappa \in (0,4]$

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D
- τ a **reverse** stopping time for η
- Pretend $\eta|_{[\tau,\infty)}$ is a GFF flow line and compute the conditional law of $\eta|_{[0,\tau]}$



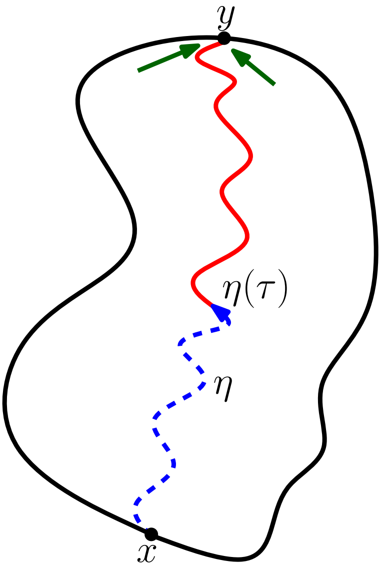
Proof of the reversibility of SLE_κ for $\kappa \in (0,4]$

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D
- τ a **reverse** stopping time for η
- Pretend $\eta|_{[\tau,\infty)}$ is a GFF flow line and compute the conditional law of $\eta|_{[0,\tau]}$
- It is $\mathrm{SLE}_\kappa(\kappa-4;\kappa-4)$ in $D_\tau \equiv D \setminus \eta([\tau,\infty))$ **conditioned** to exit at $\eta(\tau)$



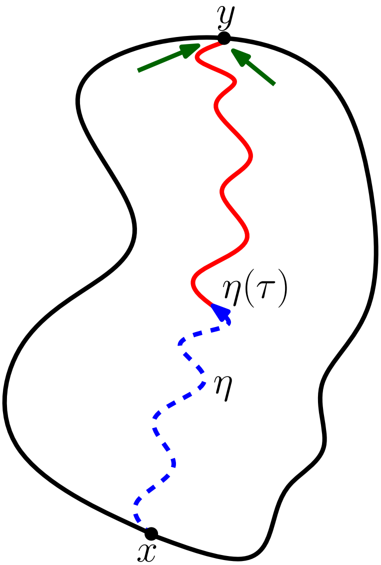
Proof of the reversibility of SLE_κ for $\kappa \in (0,4]$

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D
- τ a **reverse** stopping time for η
- Pretend $\eta|_{[\tau,\infty)}$ is a GFF flow line and compute the conditional law of $\eta|_{[0,\tau]}$
- It is $\mathrm{SLE}_\kappa(\kappa-4;\kappa-4)$ in $D_\tau \equiv D \setminus \eta([\tau,\infty))$ **conditioned** to exit at $\eta(\tau)$
- **Fact:** This implies $\eta|_{[0,\tau]}$ an SLE_κ from x to $\eta(\tau)$ in D_τ.



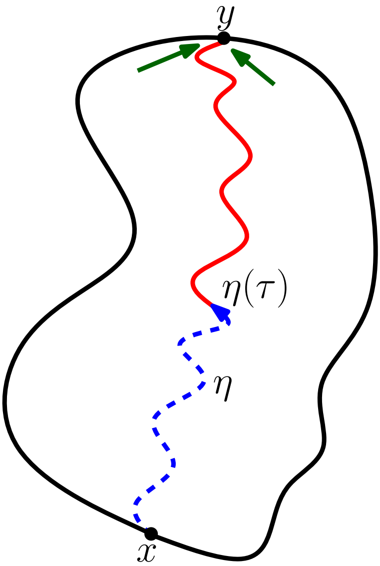
Proof of the reversibility of SLE_κ for $\kappa \in (0, 4]$

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D
- τ a **reverse** stopping time for η
- Pretend $\eta|_{[\tau,\infty)}$ is a GFF flow line and compute the conditional law of $\eta|_{[0,\tau]}$
- It is $\mathrm{SLE}_\kappa(\kappa-4;\kappa-4)$ in $D_\tau \equiv D \setminus \eta([\tau,\infty))$ **conditioned** to exit at $\eta(\tau)$
- **Fact:** This implies $\eta|_{[0,\tau]}$ an SLE_κ from x to $\eta(\tau)$ in D_τ.
- Schramm's conformal-Markov characterization implies that the time-reversal of η is an SLE_κ.



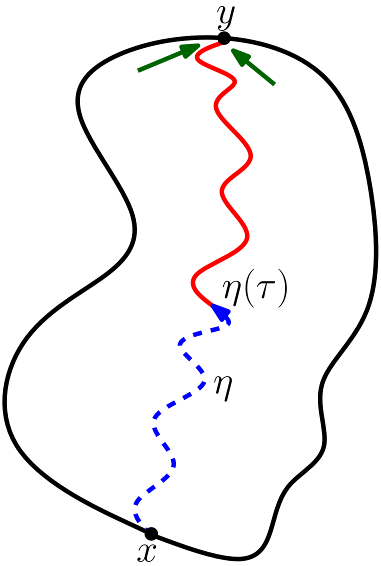
Proof of the reversibility of SLE_κ for $\kappa \in (0, 4]$

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D
- τ a **reverse** stopping time for η
- Pretend $\eta|_{[\tau,\infty)}$ is a GFF flow line and compute the conditional law of $\eta|_{[0,\tau]}$
- It is $\mathrm{SLE}_\kappa(\kappa-4;\kappa-4)$ in $D_\tau \equiv D \setminus \eta([\tau,\infty))$ **conditioned** to exit at $\eta(\tau)$
- **Fact:** This implies $\eta|_{[0,\tau]}$ an SLE_κ from x to $\eta(\tau)$ in D_τ.
- Schramm's conformal-Markov characterization implies that the time-reversal of η is an SLE_κ.
- Proof for $\mathrm{SLE}_\kappa(\rho)$ uses analogous characterization for single-force-point $\mathrm{SLE}_\kappa(\rho)$ (M., Sheffield)

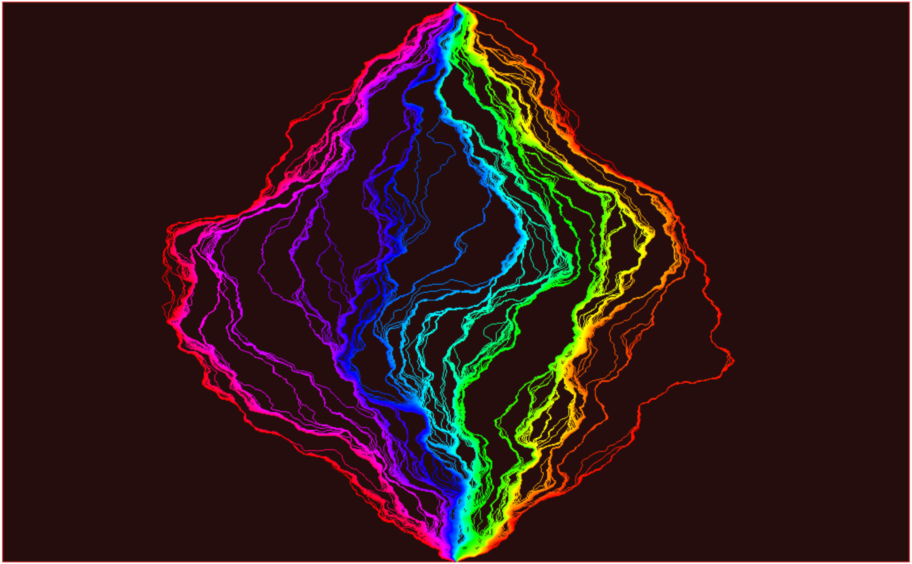


Proof of the reversibility of SLE_κ for $\kappa \in (0, 4]$

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D
- τ a **reverse** stopping time for η
- Pretend $\eta|_{[\tau,\infty)}$ is a GFF flow line and compute the conditional law of $\eta|_{[0,\tau]}$
- It is $\mathrm{SLE}_\kappa(\kappa - 4; \kappa - 4)$ in $D_\tau \equiv D \setminus \eta([\tau, \infty))$ **conditioned** to exit at $\eta(\tau)$
- **Fact:** This implies $\eta|_{[0,\tau]}$ an SLE_κ from x to $\eta(\tau)$ in D_τ.
- Schramm's conformal-Markov characterization implies that the time-reversal of η is an SLE_κ.
- Proof for $\mathrm{SLE}_\kappa(\rho)$ uses analogous characterization for single-force-point $\mathrm{SLE}_\kappa(\rho)$ (M., Sheffield)
- $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ follows using flow line tricks and a classification of "bi-chordal" SLE configurations



Corollary: the fan is "reversible"



Reversibility of SLE_κ for $\kappa \in (4,8)$

Theorem (M., Sheffield)

SLE_κ *processes are reversible for* $\kappa \in (4,8)$.

More generally, $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ *processes are reversible for* $\rho_1, \rho_2 \geq \frac{\kappa}{2} - 4$ *and are non-reversible if* $\min(\rho_1,\rho_2) < \frac{\kappa}{2} - 4$.

Reversibility of SLE_κ for $\kappa \in (4,8)$

Theorem (M., Sheffield)

SLE_κ processes are reversible for $\kappa \in (4,8)$.

More generally, $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ processes are reversible for $\rho_1, \rho_2 \geq \frac{\kappa}{2} - 4$ and are non-reversible if $\min(\rho_1,\rho_2) < \frac{\kappa}{2} - 4$.

- $\frac{\kappa}{2} - 4$ is the critical threshold for these processes to be boundary filling

Reversibility of SLE_κ for $\kappa \in (4,8)$

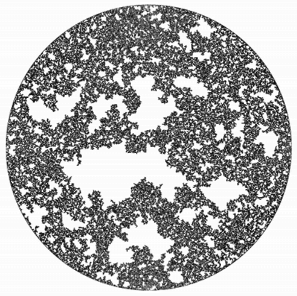
Theorem (M., Sheffield)

SLE_κ *processes are reversible for* $\kappa \in (4,8)$.

More generally, $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ *processes are reversible for* $\rho_1, \rho_2 \geq \frac{\kappa}{2} - 4$ *and are non-reversible if* $\min(\rho_1,\rho_2) < \frac{\kappa}{2} - 4$.

- $\frac{\kappa}{2} - 4$ is the critical threshold for these processes to be boundary filling

(CLE_6 simulation due to David Wilson)

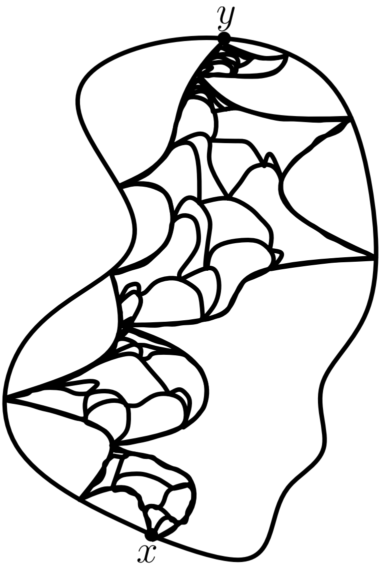


Important consequence:

- The CLE_κ processes (loop version of SLE_κ) are well defined for $\kappa \in (4,8)$.
- (Recently proved by Sheffield-Werner for $\kappa \in (8/3, 4]$ using loop soups).

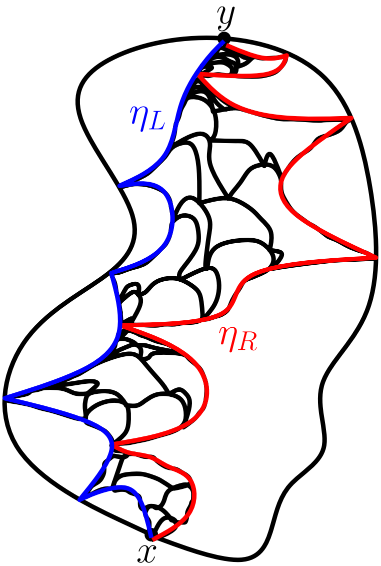
Proof of the reversibility of SLE_κ for $\kappa \in (4, 8)$, Part I

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D



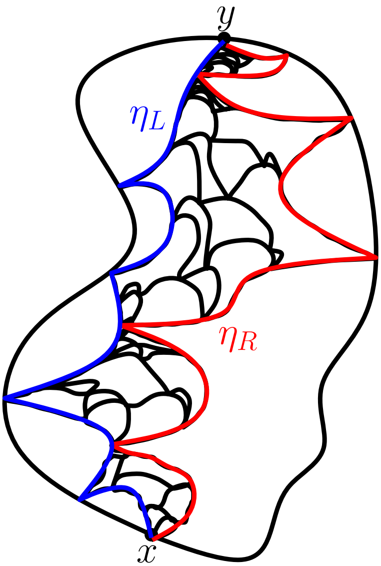
Proof of the reversibility of SLE_κ for $\kappa \in (4,8)$, Part I

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D
- **Left** and **right** boundaries of η are $\mathrm{SLE}_{16/\kappa}(\rho_1;\rho_2)$. Know how to reverse!



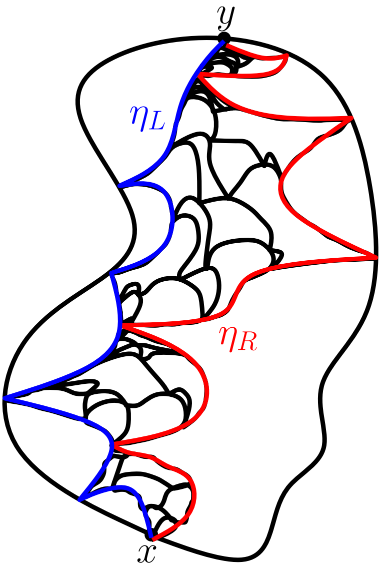
Proof of the reversibility of SLE_κ for $\kappa \in (4, 8)$, Part I

- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D
- **Left** and **right** boundaries of η are $\mathrm{SLE}_{16/\kappa}(\rho_1; \rho_2)$. Know how to reverse!
- η given η_L and η_R is a **boundary filling** $\mathrm{SLE}_\kappa(\frac{\kappa}{2} - 4; \frac{\kappa}{2} - 4)$ in each of the bubbles between η_L and η_R



Proof of the reversibility of SLE_κ for $\kappa \in (4,8)$, Part I

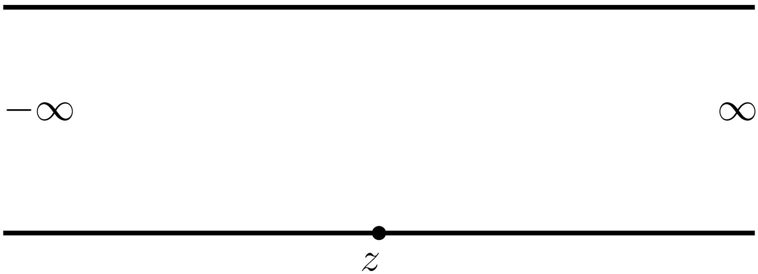
- $\eta \sim \mathrm{SLE}_\kappa$ from x to y in D
- **Left** and **right** boundaries of η are $\mathrm{SLE}_{16/\kappa}(\rho_1;\rho_2)$. Know how to reverse!
- η given η_L and η_R is a **boundary filling** $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4;\frac{\kappa}{2}-4)$ in each of the bubbles between η_L and η_R
- Suffices to show $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4;\frac{\kappa}{2}-4)$ processes are reversible



Proof of the reversibility of SLE_κ for $\kappa \in (4,8)$, Part II

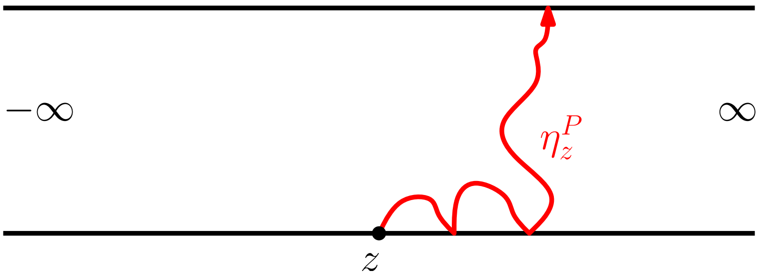
- $\eta \sim \mathrm{SLE}_\kappa(\frac{\kappa}{2} - 4; \frac{\kappa}{2} - 4)$ from ∞ to $-\infty$ in $\mathbf{R} \times [0,1]$

Proof of the reversibility of SLE_κ for $\kappa \in (4,8)$, Part II



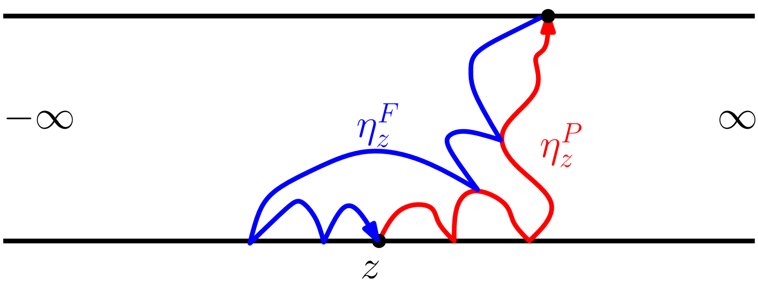
- $\eta \sim \mathrm{SLE}_\kappa(\frac{\kappa}{2} - 4; \frac{\kappa}{2} - 4)$ from ∞ to $-\infty$ in $\mathbf{R} \times [0,1]$
- Fix $z \in \mathbf{R}$; then η hits z wp 1 since η boundary filling

Proof of the reversibility of SLE_κ for $\kappa \in (4,8)$, Part II



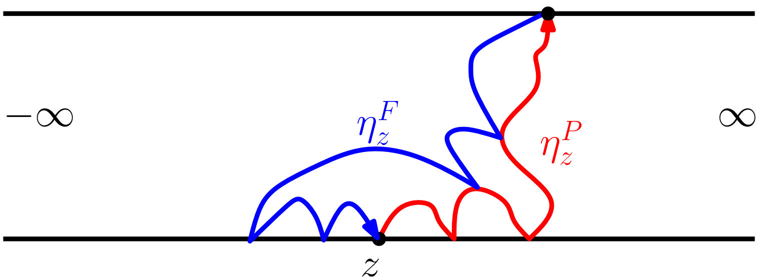
- $\eta \sim \mathrm{SLE}_\kappa(\frac{\kappa}{2} - 4; \frac{\kappa}{2} - 4)$ from ∞ to $-\infty$ in $\mathbf{R} \times [0,1]$
- Fix $z \in \mathbf{R}$; then η hits z wp 1 since η boundary filling
- η_z^P the outer boundary of η **before** hitting z;

Proof of the reversibility of SLE_κ for $\kappa \in (4,8)$, Part II



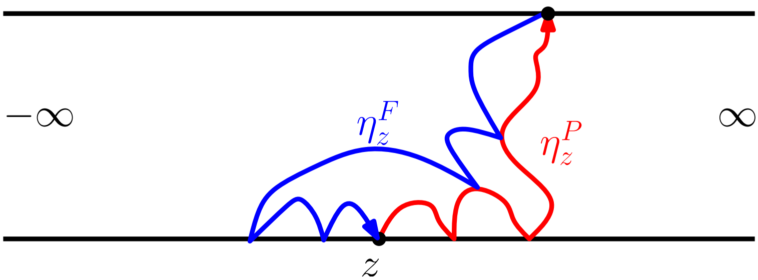
- $\eta \sim \mathrm{SLE}_\kappa(\frac{\kappa}{2} - 4; \frac{\kappa}{2} - 4)$ from ∞ to $-\infty$ in $\mathbf{R} \times [0,1]$
- Fix $z \in \mathbf{R}$; then η hits z wp 1 since η boundary filling
- η_z^P the outer boundary of η **before** hitting z; η_z^F the outer boundary **after** hitting z

Proof of the reversibility of SLE_κ for $\kappa \in (4, 8)$, Part II



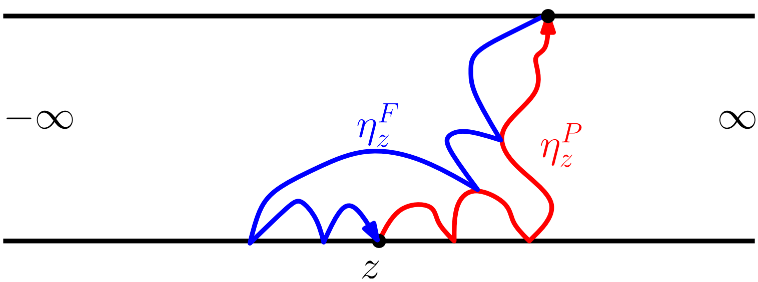
- $\eta \sim \mathrm{SLE}_\kappa(\frac{\kappa}{2} - 4; \frac{\kappa}{2} - 4)$ from ∞ to $-\infty$ in $\mathbf{R} \times [0, 1]$
- Fix $z \in \mathbf{R}$; then η hits z wp 1 since η boundary filling
- η_z^P the outer boundary of η **before** hitting z; η_z^F the outer boundary **after** hitting z
- **Main lemma:** Law of (η_z^P, η_z^F) is reflection invariant
  - Reduces to $\mathrm{SLE}_{16/\kappa}(\rho_1; \rho_2)$ reversibility by flow line tricks

Proof of the reversibility of SLE_κ for $\kappa \in (4,8)$, Part II



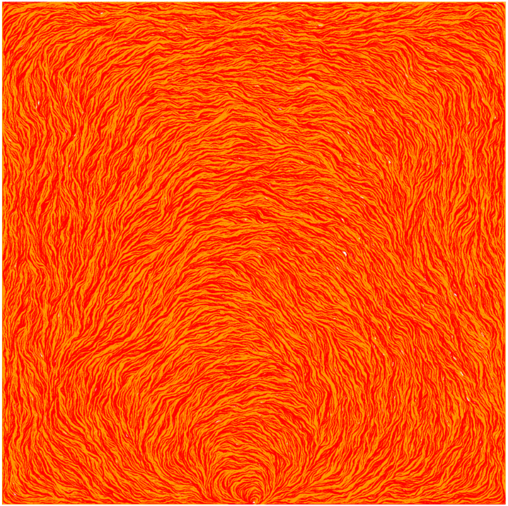
- $\eta \sim \mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$ from ∞ to $-\infty$ in $\mathbf{R} \times [0,1]$
- Fix $z \in \mathbf{R}$; then η hits z wp 1 since η boundary filling
- η_z^P the outer boundary of η **before** hitting z; η_z^F the outer boundary **after** hitting z
- **Main lemma:** Law of (η_z^P, η_z^F) is reflection invariant
  - Reduces to $\mathrm{SLE}_{16/\kappa}(\rho_1;\rho_2)$ reversibility by flow line tricks
- **Fact:** Law of η given (η_z^P, η_z^F) in each complementary component is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$

Proof of the reversibility of SLE_κ for $\kappa \in (4,8)$, Part II



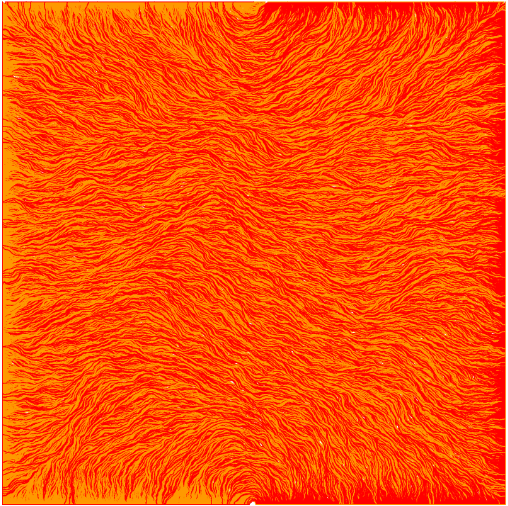
- $\eta \sim \mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$ from ∞ to $-\infty$ in $\mathbf{R} \times [0,1]$
- Fix $z \in \mathbf{R}$; then η hits z wp 1 since η boundary filling
- η_z^P the outer boundary of η **before** hitting z; η_z^F the outer boundary **after** hitting z
- **Main lemma:** Law of $(\eta_{\mathbf{z}}^{\mathbf{P}}, \eta_{\mathbf{z}}^{\mathbf{F}})$ is reflection invariant
  - Reduces to $\mathrm{SLE}_{16/\kappa}(\rho_1;\rho_2)$ reversibility by flow line tricks
- **Fact:** Law of η given $(\eta_{\mathbf{z}}^{\mathbf{P}}, \eta_{\mathbf{z}}^{\mathbf{F}})$ in each complementary component is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$
- Iterating implies that one can construct a coupling $(\eta, \widetilde{\eta})$ where $\widetilde{\eta}$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$ from $-\infty$ to ∞ so that the time-reversal of $\widetilde{\eta}$ is η

Reversibility of SLE_κ for $\kappa \geq 8$



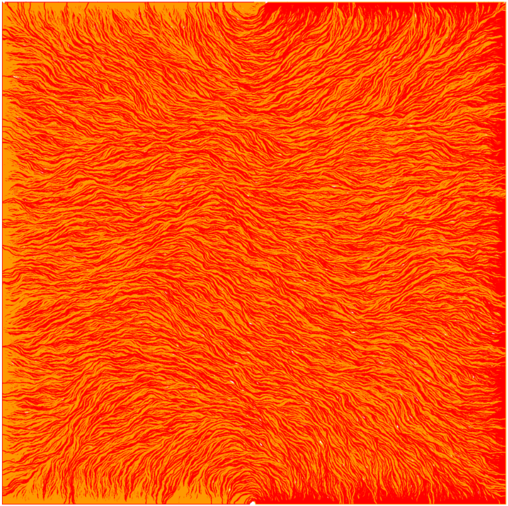
The time-reversal of an SLE_κ is not an SLE_κ for $\kappa > 8$ (Rohde-Schramm).

Reversibility of SLE_κ for $\kappa \geq 8$



Theorem (M., Sheffield) $\mathrm{SLE}_\kappa(\frac{\kappa}{4} - 2; \frac{\kappa}{4} - 2)$ is reversible for $\kappa \geq 8$.

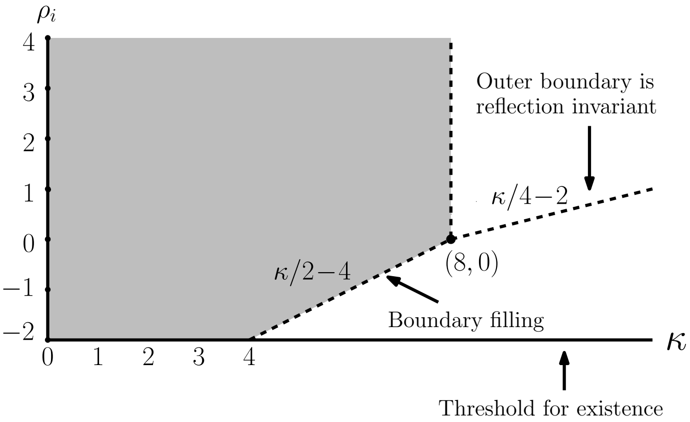
Reversibility of SLE_κ for $\kappa \geq 8$



Theorem (M., Sheffield) $\mathrm{SLE}_\kappa(\frac{\kappa}{4}-2;\frac{\kappa}{4}-2)$ is reversible for $\kappa \geq 8$. The time-reversal of ordinary SLE_κ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4;\frac{\kappa}{2}-4)$.

Summary of reversibility results for chordal SLE_κ

$\mathrm{SLE}_\kappa(\rho)$ processes are reversible when ρ is either in the grey region or dashed lines.



More reversibility results

- **Theorem** (M., Sheffield) Whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (0,4]$ and $\rho > -2$ have time-reversal symmetry.
  - Proved for $\kappa \in (0,4]$ and $\rho = 0$ by Zhan

More reversibility results

- **Theorem** (M., Sheffield) Whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (0,4]$ and $\rho > -2$ have time-reversal symmetry.
  - Proved for $\kappa \in (0,4]$ and $\rho = 0$ by Zhan
- **Idea** Use whole-plane classification of bi-chordal SLE to reduce the result to the chordal case.

More reversibility results

- **Theorem** (M., Sheffield) Whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (0,4]$ and $\rho > -2$ have time-reversal symmetry.
  - Proved for $\kappa \in (0,4]$ and $\rho = 0$ by Zhan
- **Idea** Use whole-plane classification of bi-chordal SLE to reduce the result to the chordal case.
- **Theorem** (M., Sheffield) Whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (4,8)$ and $\rho \geq \frac{\kappa}{2} - 4$ have time-reversal symmetry.
  - $\rho = \frac{\kappa}{2} - 4$ is the critical threshold where the process fills its own outer boundary

More reversibility results

- **Theorem** (M., Sheffield) Whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (0,4]$ and $\rho > -2$ have time-reversal symmetry.
  - Proved for $\kappa \in (0,4]$ and $\rho = 0$ by Zhan
- **Idea** Use whole-plane classification of bi-chordal SLE to reduce the result to the chordal case.
- **Theorem** (M., Sheffield) Whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (4,8)$ and $\rho \geq \frac{\kappa}{2} - 4$ have time-reversal symmetry.
  - $\rho = \frac{\kappa}{2} - 4$ is the critical threshold where the process fills its own outer boundary
- **Idea** Use duality and $(0,4)$ reversibility to reduce to the chordal setting

More reversibility results

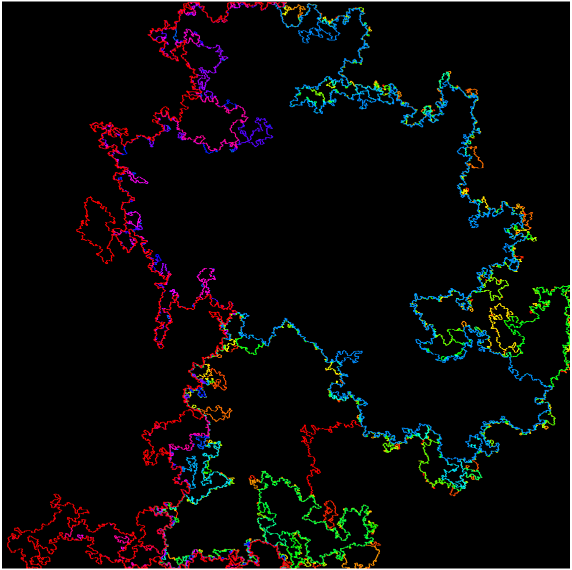
- **Theorem** (M., Sheffield) Whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (0,4]$ and $\rho > -2$ have time-reversal symmetry.
  - Proved for $\kappa \in (0,4]$ and $\rho = 0$ by Zhan
- **Idea** Use whole-plane classification of bi-chordal SLE to reduce the result to the chordal case.
- **Theorem** (M., Sheffield) Whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (4,8)$ and $\rho \geq \frac{\kappa}{2} - 4$ have time-reversal symmetry.
  - $\rho = \frac{\kappa}{2} - 4$ is the critical threshold where the process fills its own outer boundary
- **Idea** Use duality and $(0,4)$ reversibility to reduce to the chordal setting
- **Theorem** (M., Sheffield) Space-filling $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ processes for $\kappa \in (4,\infty)$ have time-reversal symmetry for $\rho \in (-2, \frac{\kappa}{2} - 2)$
  - Process is not defined for $\rho \geq \frac{\kappa}{2} - 2$

Summary

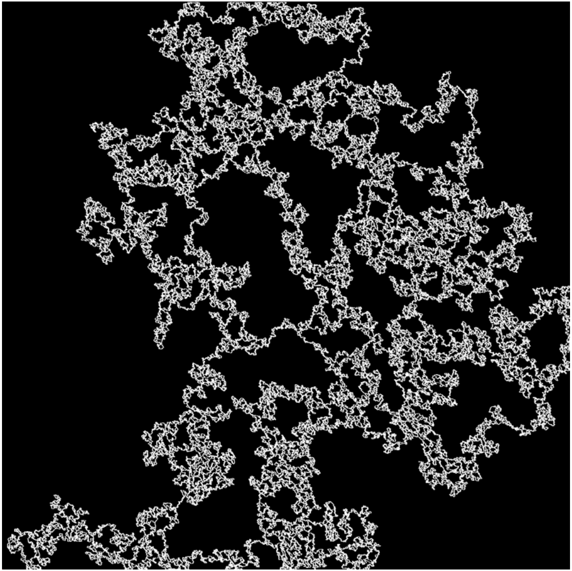
We have used the SLE/GFF coupling to establish

1. Continuity and transience (chordal, radial, whole-plane $\mathrm{SLE}_\kappa(\rho)$),
2. Duality and path decompositions (light cone, space-filling SLE),
3. Reversibility (chordal, whole-plane, and space-filling) for all κ values.

The $\mathrm{SLE}_{8/3}$ fan is reversible



SLE_6 is reversible



The SLE_6 and the $\mathrm{SLE}_{8/3}$ fan are not jointly reversible