Defining SLE in multiply connected domains using the Brownian Loop Measure

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MSRI workshop on conformal invariance and statistical mechanics

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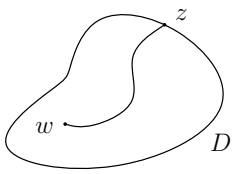
Notes taken by Samuel S Watson

We will describe desirable properties of SLE and come back to an actual definition later. One valuable feature of this perspective is that we can extend the definition to contexts other than the usual one.

We begin by setting some parameter definitions. Let $\kappa \in (0,4]$, and define $d = 1 + \kappa/8$ and $b = (6-\kappa)/(2\kappa)$ (boundary scaling exponent) and $\tilde{b} = b(\kappa-2)/4$ (bulk scaling exponent) and $c = b(3\kappa - 8)$ (central charge).



We would like to define a measure $\mu_D(z,w)$ on simple curves connecting z and w in D. These points can be anywhere in $\overline{D}$. If z is a boundary point, we insist that the boundary be smooth in a neighborhood of z. We say z is a *smooth* point if z is interior or on the boundary (and the boundary is smooth there). We define

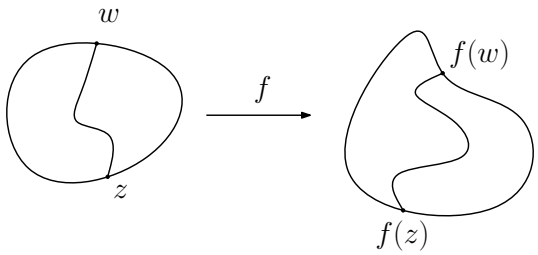
$$\Psi_D(z,w) = |\mu_D(z,w)|,$$

a (normalized) partition function. If the total mass is positive and finite, then we can define a probability measure $\mu^{\#}$ by normalization

$$\mu_D(z,w) = \Psi_D(z,w)\mu_D^{\#}(z,w),$$

We would like these measure to be supported on curves of dimension d.

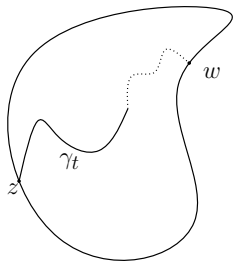
The most important property we would like μ to have is conformal covariance. The notation $\hat{b}$ means b or $\tilde{b}$.



$$f \circ \mu_D(z,w) = |f'(z)|^{\hat{b}}|f'(w)|^{\hat{b}}\mu_{f(D)}(f(z),f(w))$$

The d-dimensional length of $f \circ \gamma$ is $\int_0^{t_\gamma} |f'(\gamma(s)|^d \, ds$. This is sometimes called "natural length." It's called natural because it's what you would expect to get from discrete models by considering the number of steps and scaling appropriately. Note that the conformal covariance rule is really two properties in one: it specifies conformal *in*variance of the probability measure as well as giving the scaling rule of the partition function.

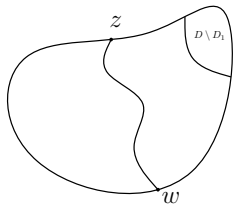
The second property we would like is the domain Markov property. We would like that $\mu_D^{\#}(z, w)$ given γ_t to be $\mu_{D\setminus\gamma_t}(\gamma(t), w)$.



Reversibility is the property that $\mu_D(z, w)$ is obtained from the reverse of $\mu_D(z, w)$.

Boundary perturbation (generalized restriction). Suppose we have $D_1 \subset D$. Then

$$\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)} = \left[e^{m_D(\gamma, D\setminus D_1)}\right]^{c/2},$$



where $m_D(\gamma, D \setminus D_1)$ is the Brownian loop measure of loops in D that intersect both γ and $D \setminus D_1$. Recall that the Brownian loop measure in D is a conformally invariant measure on collections of loops in D which satisfies the restriction property: the law of the loops in D contained in $D' \subset D$ is the Brownian loop measure in D'.

The Radon-Nikodym derivative above is conformally invariant with respect to conformal transformations of the larger domain (and not the smaller domain).

Remark for those who might appreciate it (this is not precise, but can be made so in certain special cases):

$$e^{m_D} = \frac{\frac{\det D}{\det D\setminus\gamma}}{\frac{\det D_1}{\det D_1\setminus\gamma}}$$

Relation to previous work: Oded's construction gives $\mu_D^{\#}$ in the case that D is simply connected and at most one of z, w is interior. In this case, defining

$$\Psi_D(z,w) = H_D(z,w)^b,$$

where H_D is the Green's function, gives us the remaining properties.

[Disclaimer prompted by question: this is not the only reasonable way to define SLE in multiply connected domains.]

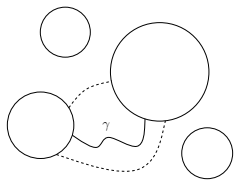
[Question: do these formulas refer to Euclidean metrics? Yes. Would it work for hyperbolic metrics too? Pretty much, yes.]

Many things are known about the natural parametrization; for example it is invariant under perturbation of the domain. It has not been proved that this parameterization is reversible.

How might one define SLE in multiply-connected domains?

Proposition 1. There is a unique extension $\Psi_D(z,w)$ satisfying all these rules.

Idea of proof. Use the Radon-Nikodym derivative. Some things come easily, like conformal covariance and the domain Markov property. What is more difficult is showing that $\Psi_D(z,w) < \infty$.



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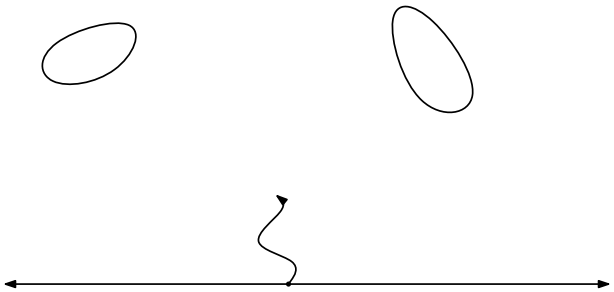
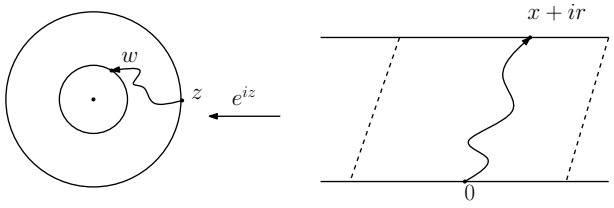
Example. Consider SLE_κ from 0 to ∞ in $\mathbb{H} \setminus \{\text{holes}\}$. We would like for the law of this path to be absolutely continuous with respect to usual SLE (without the holes). We get

$$\frac{d\mu_D}{d\mu_{\mathbb{H}}} = (\text{loop term})[\text{ratio of partition functions}],$$

where the loop term accounts for the change in the Brownian loop measure (incorporating those loops intersecting both the holes and the path). The thing we need to get this to work is smoothness of $\Psi_D(z,w)$, which is not so easy to show either.



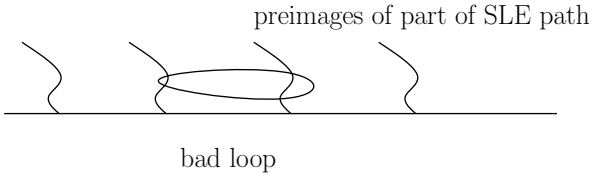
Annulus SLE (crossing) - Work by Dapeng Zhan. We'd like to define $\mu^{\#}_{A_r}(1, e^{-r+ix})$. If we could define radial SLE from an interior point to a boundary point, then annulus SLE would be the measure one obtains by conditioning on the path of an inside-out radial SLE from 0 to a positive stopping time and considering the law of the rest of the path.



Since the strip is simply connected, we can use the correspondence given by $z \mapsto e^{iz}$ between the strip and the annulus. There are a couple of issues to watch out for: some of the Brownian loops in the annulus have nonzero winding number around the inside. This just contributes an overall factor, since each of these loops hits the SLE path from the inside to the outside (this is topologically necessary). The other issue is that each loop in the annulus has infinitely many preimages in the strip. For this we have to choose a preimage for each loop and weight, on the strip side, by "bad loops" which intersect translates of the chosen preimage. Then

$$\frac{d\mu_{A_r}(z,w)}{d\mu_{S_r}(0,x+ir)}(\gamma) = \varphi(r)\Psi_{S_r}(0,x+ir)\exp\left(\frac{c}{2}m(\text{bad loops})\right).$$

It turns out that we can get smoothness in this setup, though it takes some work.



One further example: Consider radial SLE from 0 to z in the disk. Then

$$\frac{d\mu_D(0,z)}{d(\text{whole plane SLE})} = (\text{loop term})(\text{partition function ratio}),$$

where the loop term is tricky because the measure of the set of loops intersecting both $\gamma[0,\tau]$ and ∂D is infinite, so it has to be normalized appropriately.

Question: Can we treat the holes differently, as would be appropriate from a conformal field theory point of view?

Answer: Possibly, but we haven't done it.

Question: Can we define annulus SLE from z to w by just passing to the universal cover and summing the SLE paths from the preimages of z to w?

Answer: The issue is that we have to condition on the loops not self-intersecting on the annulus side. Also, we have to weight by the number of bad loops, which are ones that first intersect a translate of the curve (i.e., a different preimage) instead of the curve.