SLE, KPZ and Liouville Quantum Gravity

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Notes taken by Samuel S Watson

Recall that the Gaussian free field is a random distribution with Gaussian weight $\exp(-1/2(h, h)_\nabla]$. Another perspective is that h is a collection of Gaussian random variables $(h, f)_\nabla$ with covariance

$$\mathbb{E}(h, f_1)_\nabla (h, f_2)_\nabla = (f_1, f_2)_\nabla$$

In physics, LQG has a broader meaning, but here we take the cosmological constant equal to 0. So for us, LQG refers (heuristically) to

$$d\mu = e^{\gamma h} d^2 z,$$

where $d^2 z$ is the area measure. See the slides for an animation of what a quantum suface looks like.

We consider circle averages $h_\epsilon(z)$ (the average of the field on a circle of radius ϵ), and we will need the GFF with free boundary conditions. This is because we want to define half-circle averages $\tilde{h}_\epsilon(z)$ for $z \in \partial D$.

The variance of $h_\epsilon(z)$ is $\log(C(z, D)/\epsilon)$, where $C(z, D)$ is the conformal radius of D viewed from z. Thus

$$\mathbb{E} e^{\gamma h_\epsilon(z)} = \left(\frac{C}{\epsilon}\right)^{\gamma^2/2}$$

This observation motivates the definition of quantum area measure:

Proposition 1. $d\mu_\epsilon := \exp(\gamma h_\epsilon(z)) \epsilon^{\gamma^2/2} \, d^2 z$ converges as $\epsilon \to 0$ when $\gamma < 2$.

Proposition 2. The boundary area measure $d\tilde{\mu}_\epsilon := \exp\left(\frac{\gamma}{2} \tilde{h}_\epsilon(z)\right) \epsilon^{\gamma^2/4} \, d^2 z$ converges as $\epsilon \to 0$ when $\gamma < 2$.

We denote these two measures by $e^{\gamma h} d^2 z$ and $e^{\frac{\gamma \tilde{h}}{2}} d^2 z$.

The pictures in the slides show a picture of the random geometry in which each cell has approximately the same quantum area.

The d-dimensional Euclidean or quantum measure of planar fractal sets is characterized by scaling properties.

If $X \subset D \subset \mathbb{C}$, then the d-dimensional Euclidean fractal measure of X is multiplied by $|b|^{2-2x}$, where x is the Euclidean scaling weight, defined by $d = 2 - 2x$.

If X is a fractal subset of a random surface $S = (D, h)$ (where D is the domain in which the field lives and h is the function described before, giving the area measure). The quantum fractal measure of X is multiplied by $|b|^{2-2\Delta}$, where Δ is the quantum scaling weight.

There is a conformal invariance of Liouville Quantum Gravity. If $\psi : D \to \psi(D)$ is a conformal map, then

$$\psi(D, h) := (\psi(D), h \circ \psi^{-1} - Q \log |\psi'|,$$

where $Q = \gamma/2 + 2/\gamma$. This comes from a usual Lebesgue change of measure as well as a quantum factor to the $\gamma^2/2$.

The Knizhnik, Polyakov, Zamolodchikov (KPZ) relation

$$x = (\gamma^2/4)\Delta^2 + (1 - \gamma^2/4)\Delta$$

can equivalently be written

$$d = \alpha Q - \alpha^2/2.$$

where $\alpha = \gamma(1 - \Delta)$. This relation has several heuristic derivations due to various physicists, and it was finally proved by Scott Sheffield and BD.

Couplings between the GFF and SLE have been described by Dubedat, Sheffield, and BD. We may define the reverse SLE conformal map which sends $z \in \mathbb{H}$ to $w \in \mathbb{H} \setminus \eta$. It satisfies a certain ODE, and we think of it as "zipping up."

We may define a process $\mathfrak{h}_t(z)$ associated with a point z in the upper half-plane which gives the harmonic exposure of one side of the curve when viewed from z, plus $Q \log f_t'(z)$. This process is a martingale for a particular choice of Q.

It turns out that the conditional law of h given f_t is just another Gaussian free field in $\mathbb{H} \setminus \eta[0, t]$ with appropriate boundary conditions. To see why this works, we can find the covariation of the process $\mathfrak{h}_t(0)$ in terms of the free-boundary Green's function on $\mathbb{H}$. It turns out that

$$G_t(y, z) + \langle \mathfrak{h}_t(y), \mathfrak{h}_t(z) \rangle = G_0(y, z),$$

which motivates the equality in law of the GFF and the GFF obtained from first sampling SLE, as stated previously. The pair $(\mathbb{H}, \tilde{h} \circ f_t + \mathfrak{h}_t)$ descries the same random surface as the pair $(\mathbb{H} \setminus \eta_t, h)$. Thus the usual transformation law for h gives us a way to understand the GFF/SLE coupling.

Similarly, the conditional law of $e^{\gamma h(z)}$ given f_t is equal in law to $e^{\gamma h(w)} d^2 w$ (conformal invariance).

When $\gamma \leq 2$, the KPZ prediction $\gamma = \frac{\sqrt{25-c} - \sqrt{1-c}}{\sqrt{6}}$ for the central charge of the SLE's CFT coupled to gravity. If $\gamma > 2$, we cannot construct LQG as we have done here. However, it can be constructed, and the resulting area measure is singular (some points carry positive mass).

Conformal welding refers to gluing SLE paths together in a boundary length preserving way. Exponential martingales yield SLE quantum measures:

$$\mathbb{E}[e^{\alpha h} | f_t] = \exp\left(\alpha h - \frac{\alpha^2}{2} \langle \mathfrak{h} \rangle_t\right).$$

The expected length of an infinite SLE $\tilde{\eta}$ in D is given by

$$\nu(D) = \int_D G(z)\, d^2 z,$$

where $G(z) = |z|^a || \operatorname{Im} b|^2$.

Quantum length is defined similarly. We set

$$\nu_Q(D, h) := \int_D e^{\alpha h(z)} G(z) d^2 z,$$

where α is chosen appropriately so that there are nice martingale properties. We can think of h as having been "turned on," i.e., the previous case is just this one with $h \equiv 0$. This value of α is $\sqrt{\kappa/2}$. The exponential martingale allows us to study what happens as we "zip up" the boundary.

$$\mathbb{E}[\nu_Q(D, h)|f_t] = \int_D \mathcal{M}_t^\alpha(z) G(z)\, d^2(z),$$

which is like the formula we had before but with an extra $\mathcal{M}_t^\alpha(z)$ showing up.

Open problems:

Show that random planar graphs converge in the fine mesh limit to Liouville Quantum Gravity. We could also consider gluing along several SLE paths at the same time, and we can also think about duality, wherein we consider quantum bubbles/foam. Finally, and perhaps most famously, we can ask about the relationship between geodesics and random metrics in the discrete setting and corresponding limits in the continuum setting.

Question: What parametrization tools are used?

Answer: Only Vincent Beffara's and a couple of different papers by Lawler and collaborators.

Question: If we could define this as a metric space, what would be the Hausdorff dimension?

Answer: When $\gamma = \sqrt{8/3}$, the dimension is 4.