Liouville Quantum Gravity, KPZ & Schramm-Loewner Evolution

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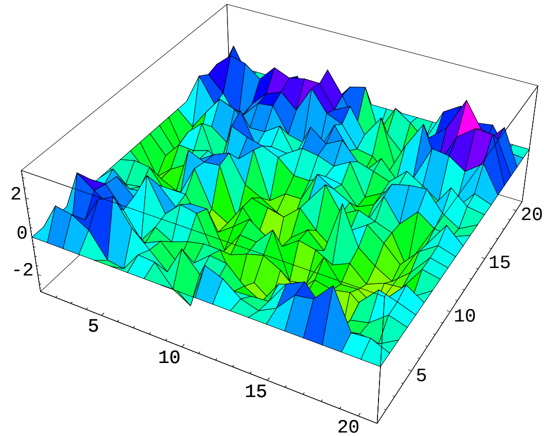
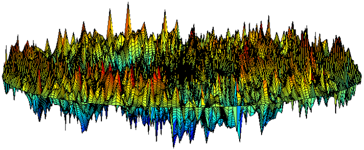
Gaussian Free Field (GFF)



Distribution h with *Gaussian weight* $\exp\left[-\frac{1}{2}(h,h)_\nabla\right]$, and **Dirichlet inner product** in domain D

$$
\begin{aligned}
(f_1, f_2)_\nabla &:= (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) d^2 z \\
&= \operatorname{Cov}\big((h, f_1)_\nabla, (h, f_2)_\nabla\big)
\end{aligned}
$$

◇ STARRING THE GFF! *(Courtesy of N.-G. Kang)* ◇



Liouville QG

Random Measure

$$d\mu = \text{``}e^{\gamma h} d^2 z\text{''}$$

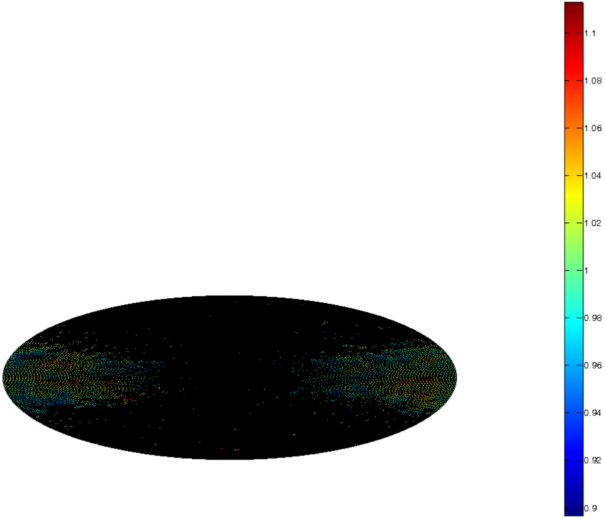
◊

The Emergence of Quantum Gravity

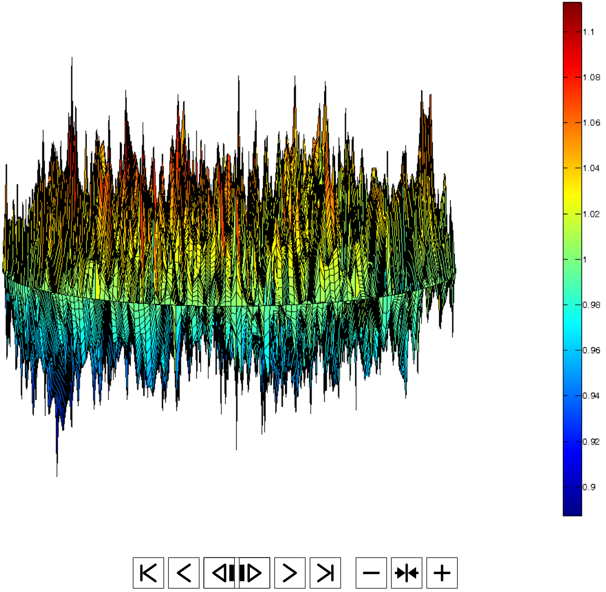
(Courtesy of N.-G. Kang)

◊

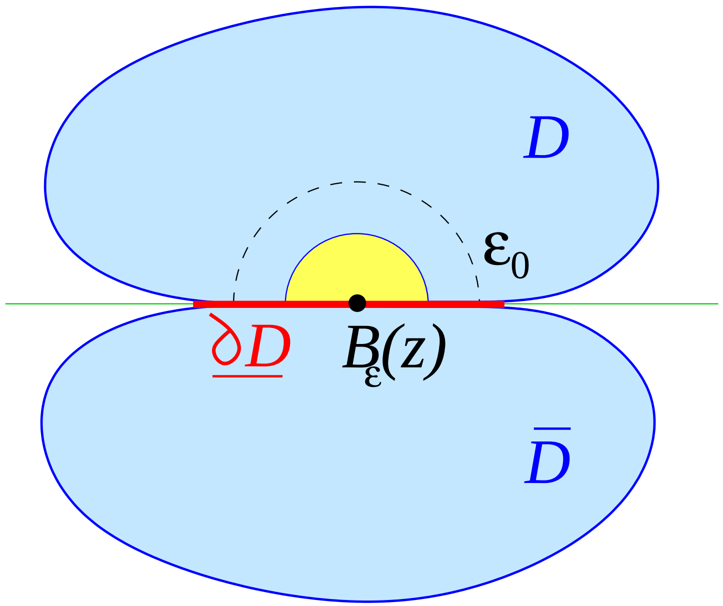
gamma = 0.02



gamma = 0.02



Bulk & Boundary Liouville Quantum Gravity



- *Circle averages* $h_\varepsilon(z)$, $z \in D$ *(Dirichlet)*
- *GFF with free boundary conditions on* ∂D
- *Half-circle averages* $\hat{h}_\varepsilon(z)$, $z \in \partial D$.

- Regularization

$h_\varepsilon(z)$ **mean value of** h **on circle** $\partial B_\varepsilon(z)$

- Variance

$$\operatorname{Var} h_\varepsilon(z) = \log[C(z,D)/\varepsilon]$$

$C(z,D)$ *conformal radius of* D *viewed from* z

$h_\varepsilon(z)$ **Gaussian random variable**

$$\mathbb{E}\, e^{\gamma h_\varepsilon(z)} = e^{\gamma^2 \operatorname{Var} h_\varepsilon(z)/2} = \left(\frac{C(z,D)}{\varepsilon}\right)^{\gamma^2/2} \qquad \square$$

Quantum Area Measure

$$d\mu_\varepsilon := \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} d^2 z$$

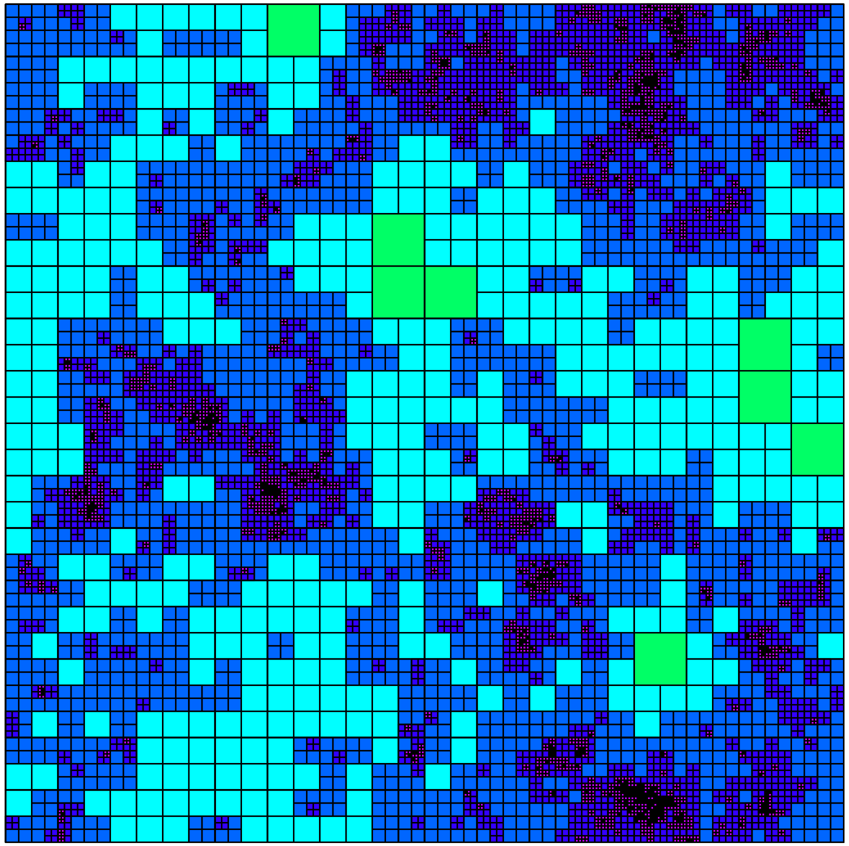
converges, as $\varepsilon \to 0$ *and for* $\gamma < 2$*, to a random measure, denoted by* $e^{\gamma h(z)} d^2 z$*.*

Quantum Boundary Measure

$$d\hat{\mu}_\varepsilon := \exp\left[\frac{\gamma}{2} \hat{h}_\varepsilon(z)\right] \varepsilon^{\gamma^2/4} dz$$

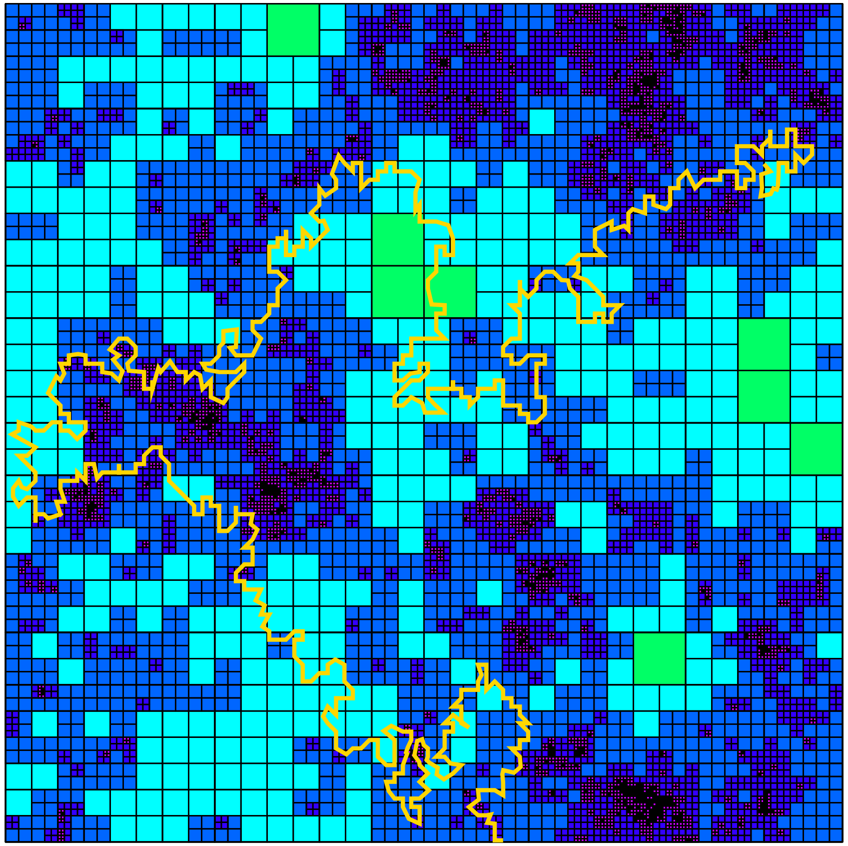
converges, as $\varepsilon \to 0$ *and for* $\gamma < 2$*, to a boundary random measure, denoted by* $e^{(\gamma/2)\,h(z)} dz$*.*

Random Surface



Euclidean squares of similar quantum area

Random Surface & Fractal Sets



Euclidean & Quantum Fractal Measures

The d-dimensional *Euclidean* or analogously *quantum measure* of planar *fractal* sets is characterized by scaling properties:

• Rescale a d-dimensional fractal $X \subset \mathcal{D} \subset \mathbb{C}$ via the map $z \to \psi(z) = bz$, $b \in \mathbb{C}$ (so that the Euclidean area of the domain $\mathcal{D}$ is multiplied by $|b|^2$); then the *d-dimensional Euclidean fractal measure* of X is multiplied by $|b|^d = |b|^{2-2x}$, where x (the *Euclidean scaling weight*) is defined by $d := 2 - 2x \,(\leqslant 2)$.

• If X is a *fractal subset* of a *random surface* $\mathcal{S} := (\mathcal{D}, h)$, and we rescale $\mathcal{S}$ so that its quantum area increases by a factor of $|b|^2$, then the *quantum fractal measure* $\mathcal{Q}(X, h)$ of X is multiplied by $|b|^{2-2\Delta}$, where Δ is the analogous *quantum scaling weight*.

- $\mathcal{Q}\big(\psi(X,h)\big) = \mathcal{Q}(X,h)$ whenever ψ is conformal and

$$\begin{aligned}\psi(\mathcal{D},h) &:= \big(\psi(\mathcal{D}), h\circ\psi^{-1} - Q\log|\psi'|\big)\\ Q &:= \frac{\gamma}{2}+\frac{2}{\gamma}.\end{aligned}$$

The pair $\mathcal{S} = (\mathcal{D},h)$ describes the same Liouville quantum surface (up to coordinate change) as the conformally transformed pair $\psi(\mathcal{D},h)$.

- The *Knizhnik, Polyakov, Zamolodchikov (KPZ)* relation

$$x = (\gamma^2/4)\Delta^2 + \big(1-\gamma^2/4\big)\,\Delta$$

is then equivalent to

$$d = \alpha Q - \alpha^2/2,$$

where $d = 2-2x$ and $\alpha := \gamma(1-\Delta)$.

SLE - GFF (QG) Coupling

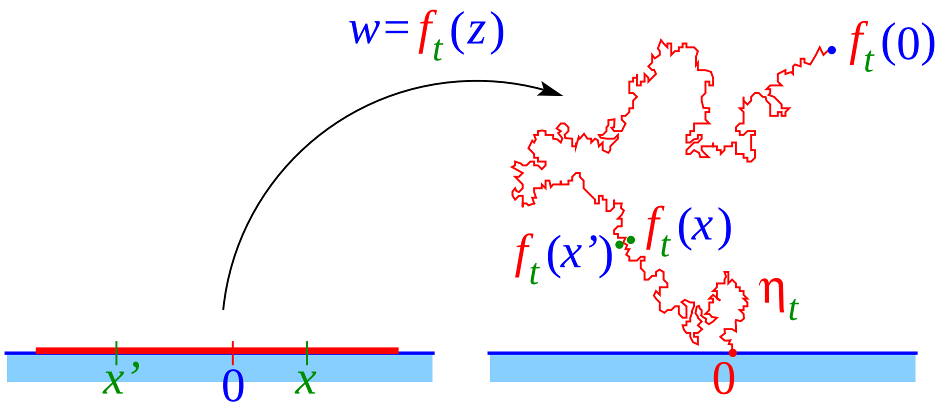
Dubédat, 2009

Sheffield, arXiv:1012.4797

D. & Sheffield, PRL **107***, 131305 (2011), arXiv:1012.4800*

Miller & Sheffield, arXiv:1201.14896-98

"Zipping-up" SLE Map



Let f_t be the (reverse) SLE_κ conformal map

$$z \in \mathbb{H} \to w = f_t(z) \in \mathbb{H} \setminus \eta_t,$$

with trace η_t and tip $f_t(0)$ [$t = 0$, $f_0(z) = z$]. It satisfies the stochastic differential equation (B_t *standard Brownian motion*)

$$d\,f_t(z) = -2dt/f_t(z) - \sqrt{\kappa}\,dB_t.$$

(Reverse) SLE Martingale

Real stochastic process in the upper-half plane:

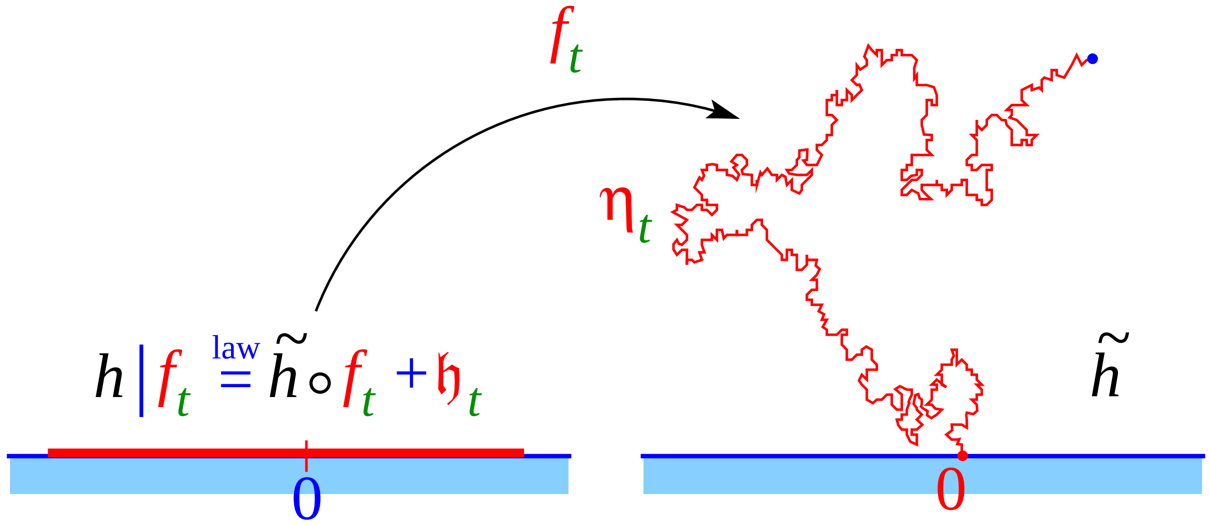
$$\begin{aligned} \mathfrak{h}_0(z) &:= \frac{2}{\sqrt{\kappa}} \log |z|, \\ \mathfrak{h}_t(z) &:= \mathfrak{h}_0 \circ f_t(z) + Q \log |f_t'(z)|. \end{aligned}$$

This process $\mathfrak{h}_t(z)$ is a *martingale* (so that $\mathbb{E}\mathfrak{h}_t(z) = \mathfrak{h}_0(z)$) for the particular choice:

$$Q = \sqrt{\kappa}/2 + 2/\sqrt{\kappa},$$

for which $d\mathfrak{h}_t(z) = -\Re[2/f_t(z)]dB_t$.

SLE–GFF Coupling

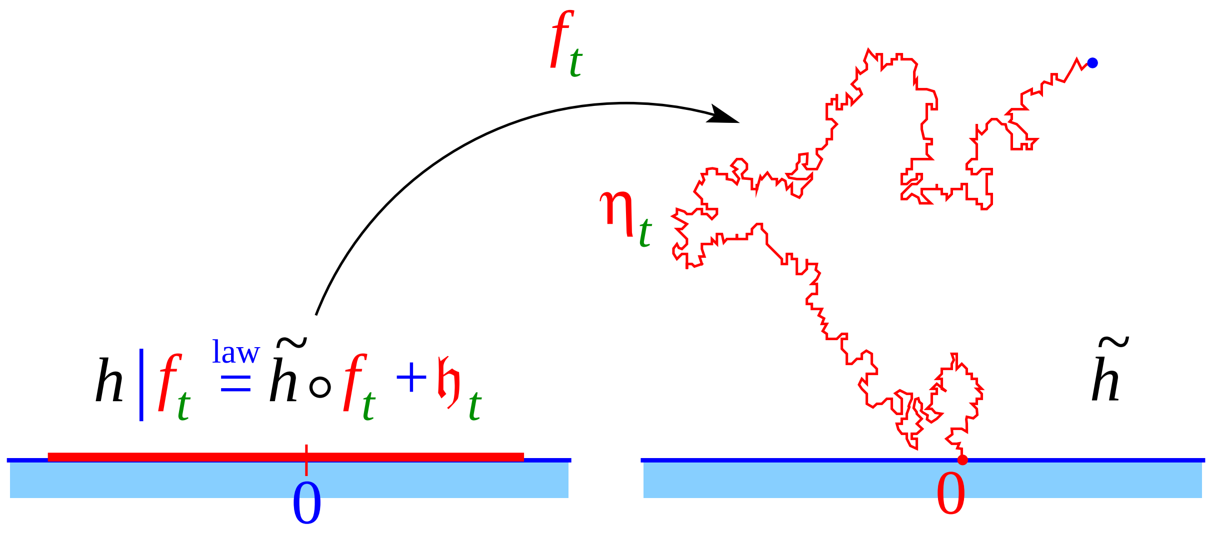


Define $h := \tilde{h} + \mathfrak{h}_0$, sum of the GFF $\tilde{h}$ on $\mathbb{H}$ with *free boundary conditions* on $\mathbb{R}$, and of the deterministic function $\mathfrak{h}_0$. Given f_t, the conditional law of h (denoted by $h|\,f_t$) is

$$h(z)|\,f_t \overset{\text{(law)}}{=} \tilde{h} \circ f_t(z) + \mathfrak{h}_t(z),$$

where $\tilde{h} \circ f_t$ is the pullback of the free boundary GFF $\tilde{h}$.

SLE–GFF Coupling



$$h(z)\,|\,f_t \overset{\text{(law)}}{=} \tilde{h} \circ f_t(z) + \mathfrak{h}_t(z)$$

To sample h, one can first sample the B_t process (which determines f_t), then sample independently the f.b.c. GFF $\tilde{h}$ and take the above sum *[Sheffield, 2010].*

The conditional expectation w.r.t. $\tilde{h}$ is the *martingale* $\mathbb{E}\left[h(z)\,|\,f_t\right] = \mathfrak{h}_t(z)$.

Neumann Green function

Consider the *Neumann Green function* in $\mathbb{H}$, $G_0(y,z) := -\log(|y-z||y-\bar{z}|)$, and define the *time-dependent* $G_t(y,z) := G_0\big(f_t(y), f_t(z)\big)$, i.e., G_0 taken at image points under f_t. A calculation of the Green function's variation shows that $-dG_t(y,z) = d\langle \mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle$ (*Hadamard's formula*). Integrating w.r.t. t yields the covariation of the $\mathfrak{h}_t$ martingales

$$\langle \mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle = G_0(y,z) - G_t(y,z).$$

In the $y \to z$ limit:

$$\langle \mathfrak{h}_t(z), \mathfrak{h}_t(z)\rangle = C_0(z) - C_t(z),$$

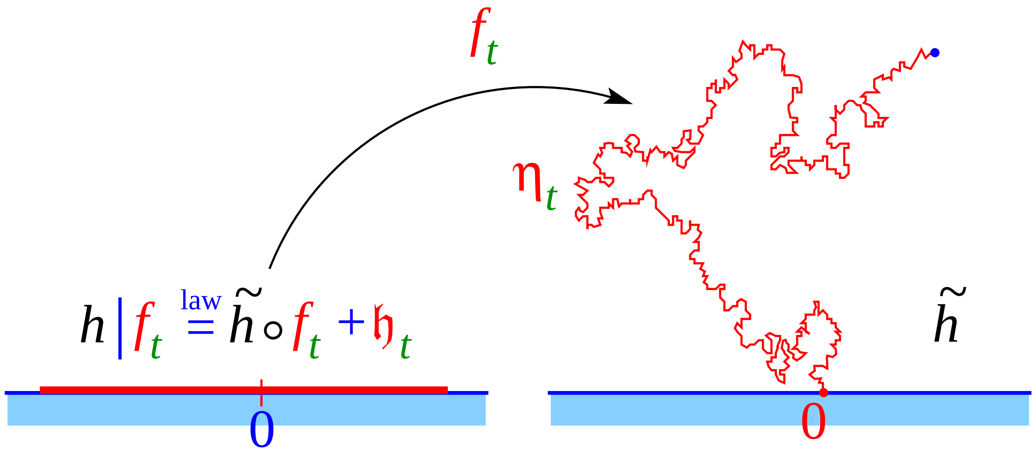
where $C_t(z) := -\log\left[\Im f_t(z) | f_t'(z)|\right]$.

SLE–GFF Coupling

Define the *covariance*: $\mathrm{Cov}[A,B] := \mathbb{E}[AB] - \mathbb{E}[A]\mathbb{E}[B]$.
Recall that the Green's function $G_0(y,z) = \mathrm{Cov}[\tilde{h}(y), \tilde{h}(z)]$, thus $G_t(y,z) = \mathrm{Cov}[\tilde{h} \circ f_t(y), \tilde{h} \circ f_t(z)]$. *The random distribution $\tilde{h} \circ f_t$ and the set of (time changed) Brownian motions $\mathfrak{h}_t$ are Gaussian processes, whose respective covariance G_t and covariation $\langle \mathfrak{h}_t, \mathfrak{h}_t \rangle$ thus add to constant G_0:*

$$
\begin{aligned}
G_t(y,z) + \langle \mathfrak{h}_t(y), \mathfrak{h}_t(z) \rangle &= G_0(y,z) \\
\mathrm{Cov}[\tilde{h} \circ f_t(y), \tilde{h} \circ f_t(z)] + \langle \mathfrak{h}_t(y), \mathfrak{h}_t(z) \rangle &= \mathrm{Cov}[\tilde{h}(y), \tilde{h}(z)] \\
&= \mathrm{Cov}[h(y), h(z)] \quad \square
\end{aligned}
$$

Liouville Invariance



Recall that $h := \tilde{h} + \mathfrak{h}_0$, and $\mathfrak{h}_t := \mathfrak{h}_0 \circ f_t + Q\log|f_t'|$. Hence $\tilde{h}\circ f_t + \mathfrak{h}_t = h \circ f_t + Q\log|f_t'|$. For $Q = \gamma/2 + 2/\gamma$, this is the transformation law of the GFF h under the conformal map f_t^{-1}. The pair $(\mathbb{H}, \tilde{h}\circ f_t + \mathfrak{h}_t) = f_t^{-1}(\mathbb{H}\setminus\eta_t, h)$ describes the same random surface as the pair $(\mathbb{H}\setminus\eta_t, h)$: *Given f_t, the image under f_t of the measure $e^{\gamma h(z)}d^2z$ in $\mathbb{H}$ is a random measure whose law is the a priori (unconditioned) law of $e^{\gamma h(w)}d^2w$ in $\mathbb{H}\setminus\eta_t$.*

Liouville Quantum Measure

$$\left(e^{\gamma h(z)} \,|\, f_t\right) d^2z \stackrel{\text{(law)}}{=} e^{\gamma h(w)}\, d^2w \quad \text{(conformal invariance)}$$

for $d = 2 = \gamma Q - \gamma^2/2$, *i.e.*, $Q = \gamma/2 + 2/\gamma = \sqrt{\kappa}/2 + 2/\sqrt{\kappa}$

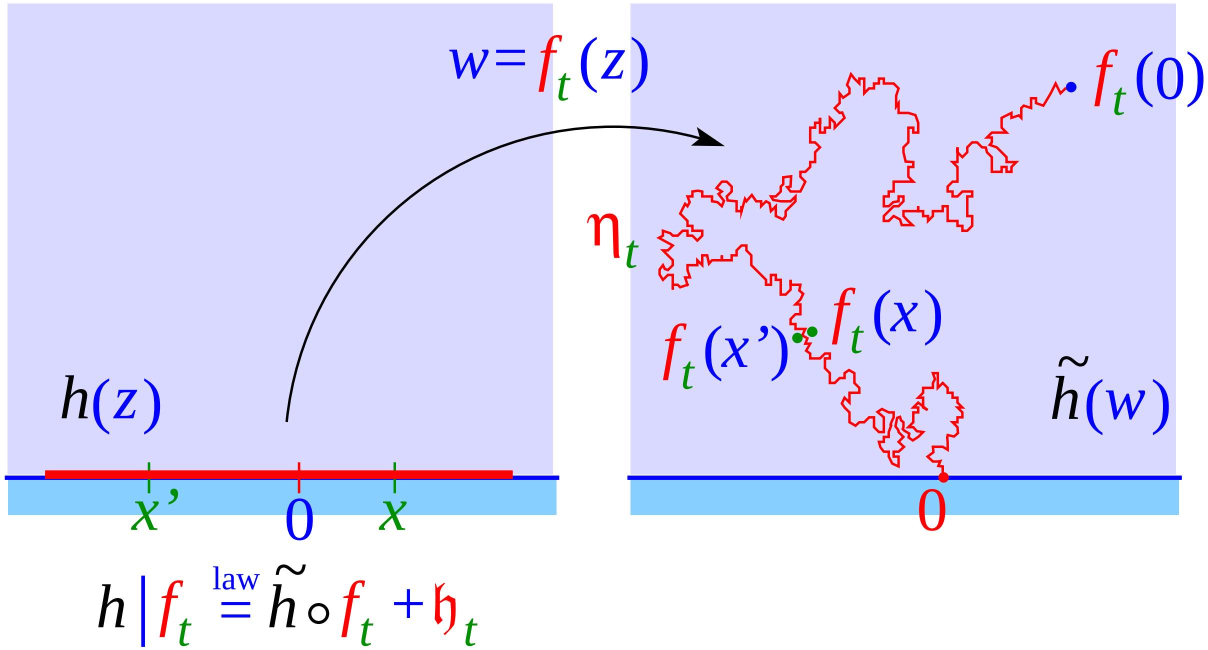
$$\Longrightarrow \ \gamma = \sqrt{\kappa} \wedge \left(4/\sqrt{\kappa}\right), \ \ \gamma' = 4/\gamma$$

• $\gamma \leqslant 2$: *KPZ prediction* $\gamma = \left(\sqrt{25-c} - \sqrt{1-c}\right)/\sqrt{6}$ for the *central charge* $c = \frac{1}{4}(6-\kappa)(6-16/\kappa) \leqslant 1$ of the SLE's CFT coupled to gravity.

• $\gamma' = 4/\gamma > 2$: *Duality* property of Liouville quantum gravity; the quantum measure develops atoms with localized area.

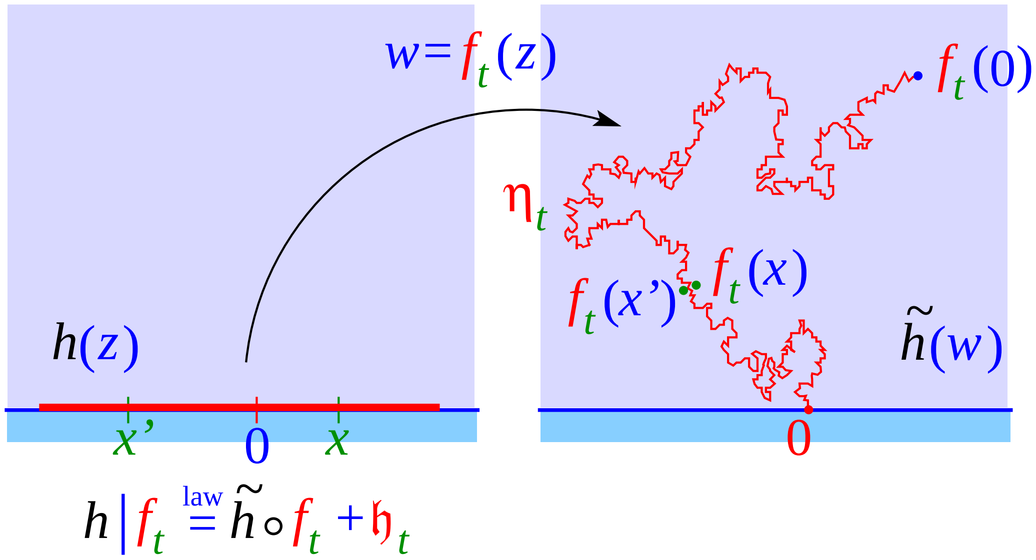
Conformally welding two γ*-Liouville quantum surfaces produces* SLE$_\kappa$.

Conformal Welding



Conformal welding: the *quantum boundary lengths* of any pair of real segments $[0,x]$ and $[x',0]$ such that $f_t(x) = f_t(x')$ on the SLE trace are *a.s. equal* for $h = \tilde{h} + \mathfrak{h}_0$ *[Sheffield, 2010].*

Liouville Quantum Gravity & SLE



- *Exponential martingales yield SLE quantum measures:*

$$\mathbb{E}\left[h|f_t\right] = \mathfrak{h}_t, \quad \mathbb{E}\left(e^{\alpha h}|f_t\right) = \exp\left[\alpha\mathfrak{h}_t - (\alpha^2/2)\langle \mathfrak{h}_t, \mathfrak{h}_t\rangle\right]$$

[D. & Sheffield, PRL **107***, 131305 (2011)]*

SLE Exponential Martingales & KPZ

$$\mathcal{M}_t^{\alpha}(z) := \mathbb{E}\left(e^{\alpha h(z)} | f_t\right), \ \alpha \in \mathbb{R}$$

$$\left(e^{\alpha h(z)} | f_t\right) d^2 z \stackrel{(\text{law})}{=} |f_t'(z)|^{d-2} e^{\alpha h(w)} d^2 w$$

$$d := \alpha Q - \alpha^2/2 \ \ (\text{KPZ})$$

where $w = f_t(z)$, $d^2 w = |f_t'(z)|^2 d^2 z$.

SLE Exponential Martingales

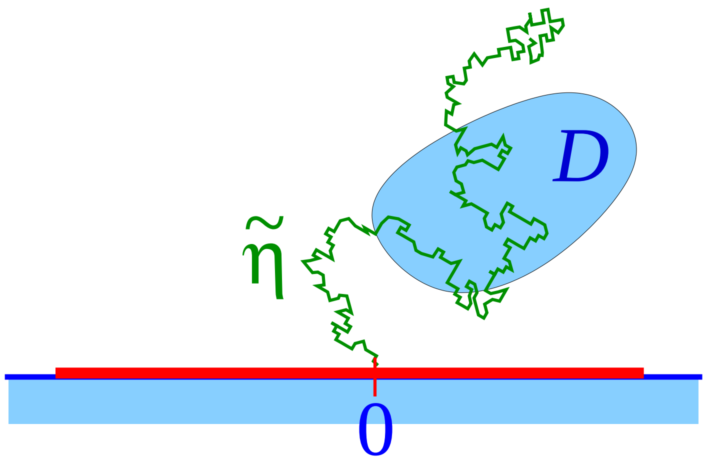
- Conditional expectation w.r.t. GFF h: $\mathbb{E}\left[h(z)\,|\,f_t\right] = \mathfrak{h}_t(z)$.
- Conditional expectations of exponentials:

$$
\begin{aligned}
\mathcal{M}_t^{\alpha}(z) &:= \mathbb{E}\left(e^{\alpha h(z)}\,|\,f_t\right), \;\; \alpha \in \mathbb{R} \\
&= \exp\left[\alpha \mathfrak{h}_t(z) - (\alpha^2/2) C_t(z)\right] \\
&= \left|f_t'(z)\right|^d |w|^{2\alpha/\sqrt{\kappa}} (\Im w)^{-\alpha^2/2}; \; d := \alpha Q - \alpha^2/2 \\
C_t(z) &:= \langle \mathfrak{h}_t(z), \mathfrak{h}_t(z)\rangle = \log[\Im f_t(z)\,|f_t'(z)|]
\end{aligned}
$$

where $w = f_t(z)$; $\mathcal{M}_t^{\alpha}(z)$ *is an exponential martingale with respect to the Brownian motion driving the SLE process:*

$$
\mathbb{E}\mathcal{M}_t^{\alpha}(z) = \mathcal{M}_0^{\alpha}(z) = |z|^{2\alpha/\sqrt{\kappa}} (\Im z)^{-\alpha^2/2}.
$$

SLE Natural Length



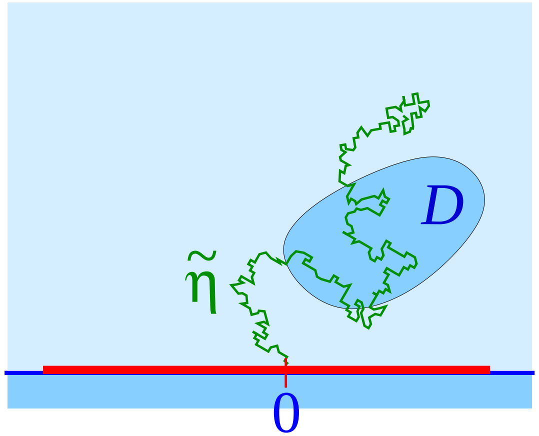
Expected (w.r.t. the $\mathrm{SLE}_{\kappa\in[0,8]}$ law) *length* of an infinite SLE $\tilde{\eta}$ in D *(Lawler & Sheffield, 2009)*

$$\nu(D) = \int_D G(z)\, d^2 z,$$

SLE Green's function in $\mathbb{H}$:

$$G(z) := |z|^a |\Im z|^b, \;\; a = 1 - 8/\kappa, \;\; b = 8/\kappa + \kappa/8 - 2.$$

SLE Quantum Length



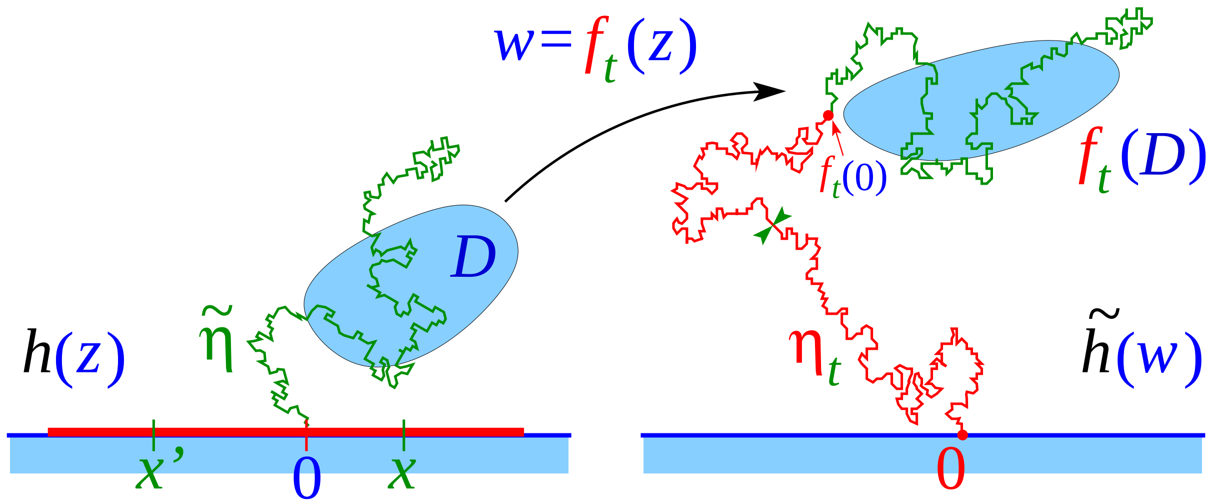
$h = \tilde{h} + \mathfrak{h}_0$

Expected (w.r.t. $\tilde{\eta}$, given h) Liouville *quantum length* ν_Q in D

$$\nu_Q(D,h) := \int_D e^{\alpha h(z)} G(z) d^2 z,$$

$\alpha = \sqrt{\kappa}/2$ ($= \gamma/2$ for $\kappa \leqslant 4$, and $\gamma'/2$ for $\kappa > 4$) satisfies KPZ for the SLE Hausdorff dimension $d = 1 + \kappa/8$.
[Doob-Meyer, second moment method.]

Expected SLE Quantum Length



$$h\,\big|\,f_t \overset{\text{law}}{=} \tilde{h}\circ f_t + \mathfrak{h}_t$$

$$\mathbb{E}[\nu_Q(D,h)\,|\,f_t] \;=\; \int_D \mathcal{M}_t^{\alpha}(z)G(z)d^2z$$

$$\mathbb{E}\nu_Q(D,h) \;=\; \int_D \mathcal{M}_0^{\alpha}(z)G(z)d^2z = \int_D (\sin\vartheta)^{8/\kappa-2}d^2z,$$

with $\vartheta := \arg z$. It is finite for $\kappa \in [0,8)$ and coincides with the *Euclidean area* of D for $\kappa = 4$.

PERSPECTIVES

- *Scaling limits of discrete models on random planar graphs*
- *Quantum wedges and cones*
- *Quantum bubbles and foam* ($\gamma\gamma' = 4$ *duality*)
- *Geodesics & random metrics*

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