Hausdorff dimension of the CLE gasket

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For motivation, we discuss the $O(n)$ loop model. Consider a finite hexagon graph with faces colored black or white. We define a model for which a configuration ω has $\mathbb{P}(\omega) = n^{\ell(\omega)}x^{e(\omega)}$. A conformally invariant scaling limit is conjectured when $x = x_c = 1/\sqrt{2+\sqrt{2-n}}$.

What should the limit of such a loop configuration be?

Well, if we take all black faces on one boundary arc and all white faces on the complement, then we should get a chordal $\mathsf{SLE}(\kappa)$ curves.

The analogous construction with monochromatic boundary conditions is the conformal loop ensemble, which is a random collection of non-crossing loops. The extreme values of κ for CLE_κ are 8/3 and 8. The gasket is the set of all points surrounded by no loops.

Theorem 1. The Hausdorff dimension of the CLE gasket is given by

$$2 - \frac{(8-\kappa)(3\kappa-8)}{32\kappa}.$$

This theorem was proved in several parts, with the upper bound coming from one paper, and the matching lower bounds come from two different papers with different techniques for different ranges of κ.

The fractal dimension of each loop is $1 + \frac{\kappa}{8}$, so the gasket dimension has to be larger (and in fact is strictly larger) than this expression.

We may define a discrete exploration process that traces out the loops in an $O(n)$ model, targeted toward a fixed point v in the graph. The analogous construction in the continuum setting is a bit tricky because chordal SLE is not defined from a boundary point to the same boundary point. To handle this, we introduce a coordinate change which gives us a "force point," and makes our exploration process an $\text{SLE}_\kappa(\kappa-6)$. The angle θ_t between the driving function and the force point evolves according to the SDE

$$d\theta_t = \sqrt{\kappa}dB_t + \frac{\kappa-4}{2}\cot(\theta_t/2)dt,$$

when θ_t is not at an integer multiple of 2π. We define the $\text{SLE}_\kappa(\kappa-6)$ process using an instantaneously reflected version of this diffusion.

We can couple together a collection of these branching processes targeted at all the points in the domain, and these processes agree until the first time at which their targets are separated. We use this exploration tree to form the loops in our CLE loop model.

CLE_κ for $\kappa \leq 4$ has an equivalent definition as the outer boundary of a Brownian loop soup (Sheffield and Werner, 2010), and the $SLE_\kappa(\kappa - 6)$ processes are continuous curves (Miller and Sheffield, 2012).

The probability that z is within ϵ of the gasket is asymptotic to ϵ^α, where $\alpha(\kappa)$ is the fraction given in the original theorem statement. This gives an upper bound by a first-moment method.

For a lower bound, we use a Hausdorff dimension "black box" theorem which involves a second-moment hypothesis. This hypothesis essentially asserts that two events E_z and E_w are correlated down to scale $|z - w|$ and independent for subsequent scales. This comes naturally if we have some tree structure for our events.

The challenge is how to define the events E_z^j, and how to obtain a tree structure. The key idea is to use the clockwise loops rather than the counterclockwise loops (in the exploration tree) to define the tree structure. The event E_1^0 is that we reach $e^{-\beta+\Delta}$ without forming a CCW loop around 0, and then make a clockwise loop before reaching distance $e^{-\beta}$. To estimate the probability of this, we uniformize by g_{τ_1}, where τ_1 is the first time we hit $e^{-(\beta-\delta)}$.

To estimate what happens in Stage 2, we consider some idealized curve going around the annulus, and figure out that there's a positive probability that the driving function stays near the driving function corresponding to the idealized curve.

Finally, we need to show that there's a positive probability that the loop is closed once it gets close to being closed. This follows from Beurling estimates.