Hausdorff dimension of the CLE gasket

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Microsoft Research   Stanford University

MSRI   27 March 2012

Outline

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1. The $O(n)$ model
2. Hausdorff dimension of the CLE gasket
3. Exploring a CLE
4. Ideas for the lower bound
5. An SLE estimate

The $O(n)$ model

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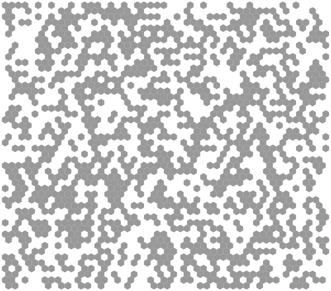

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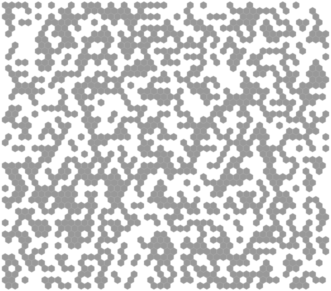

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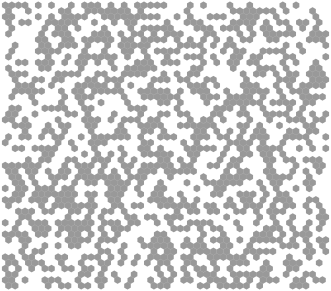

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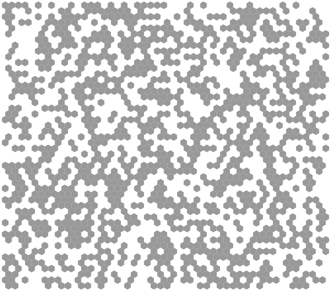

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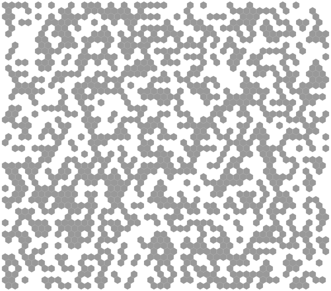

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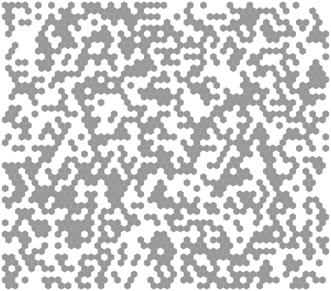

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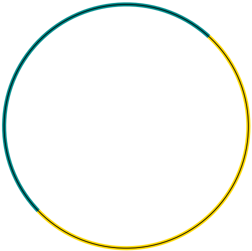
Chordal $O(n)$ model:
$O(n)$ model with Dobrushin boundary conditions



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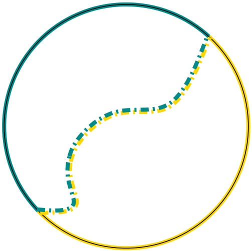

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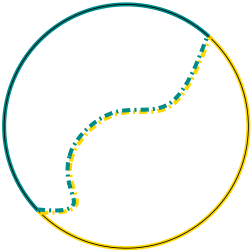

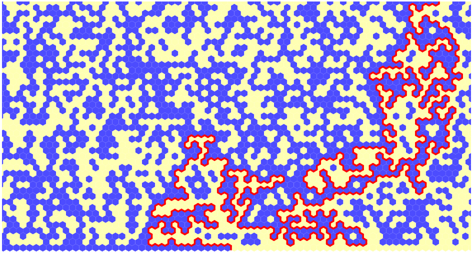

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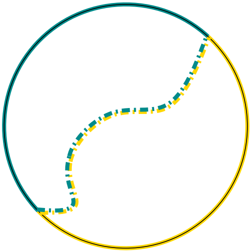

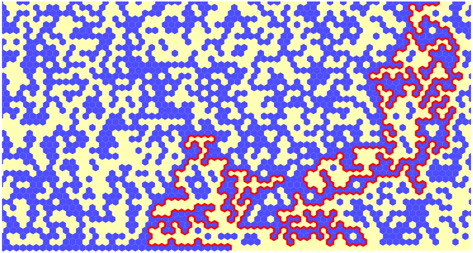

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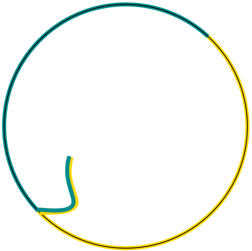

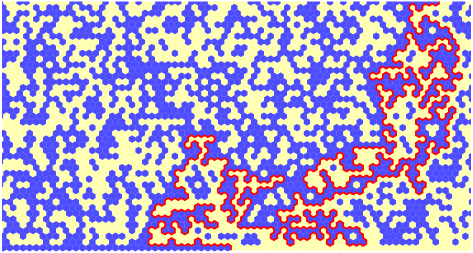

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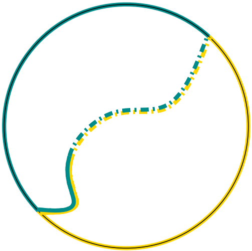

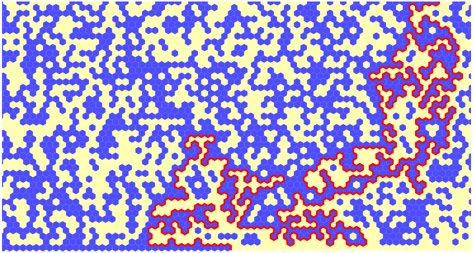

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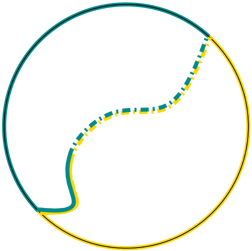

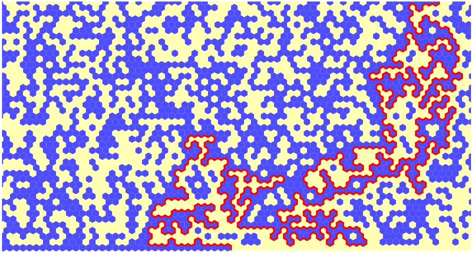

Has **domain Markov property**
Scaling limit of chordal arc should be **chordal** SLE_κ

A short review of SLE: [Schramm IJM '00]

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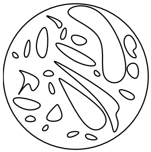
[Sheffield Duke '09, Werner CRM '03, Sheffield–Werner Annals (to appear)]

Conformal loop ensemble CLE_κ:



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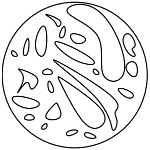

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Conformal loop ensemble CLE_κ:
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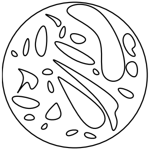

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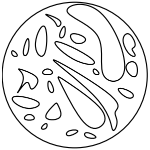

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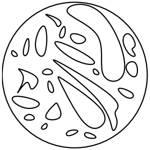

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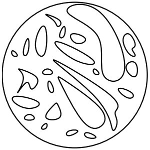

Canonical scaling limit of discrete loop ensembles
Loops of CLE_κ look locally like SLE_κ
$\mathrm{CLE}_{8/3}$ is empty;

Conformal loop ensembles

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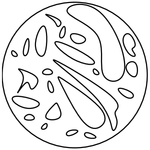

Canonical scaling limit of discrete loop ensembles
Loops of CLE_κ look locally like SLE_κ
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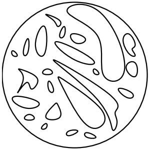

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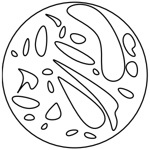

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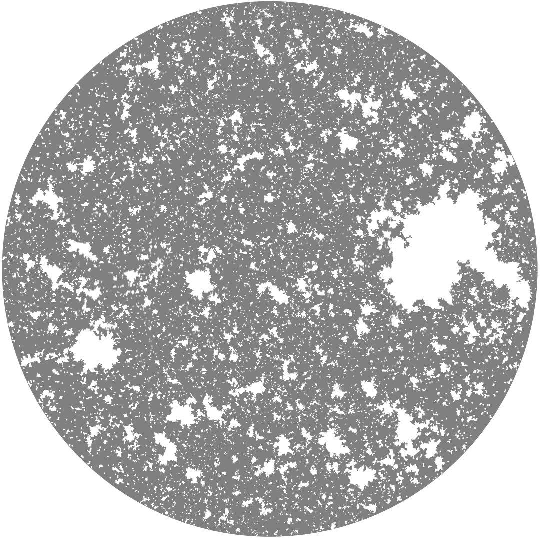
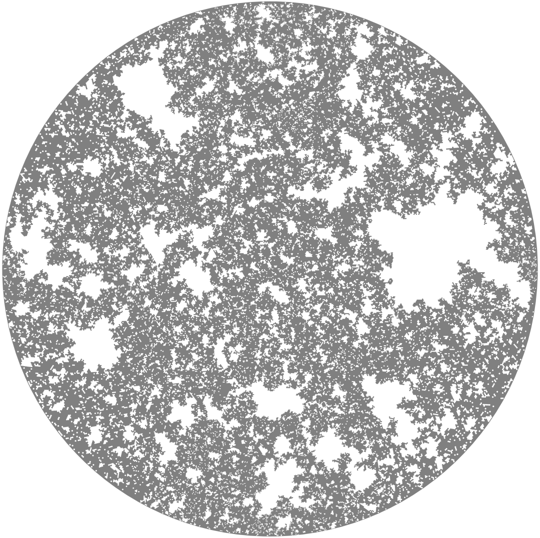


$\kappa = 3$ [figure: David Wilson]



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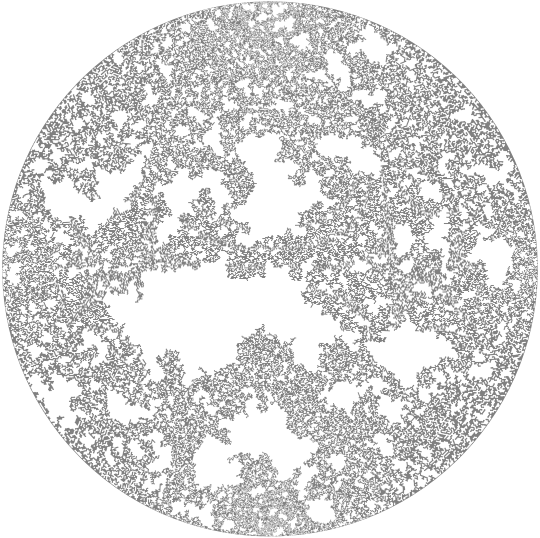

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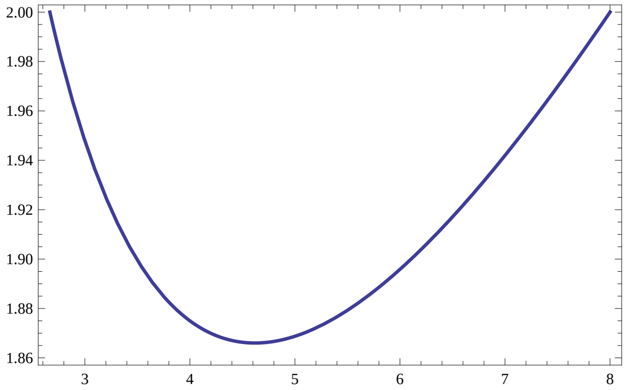
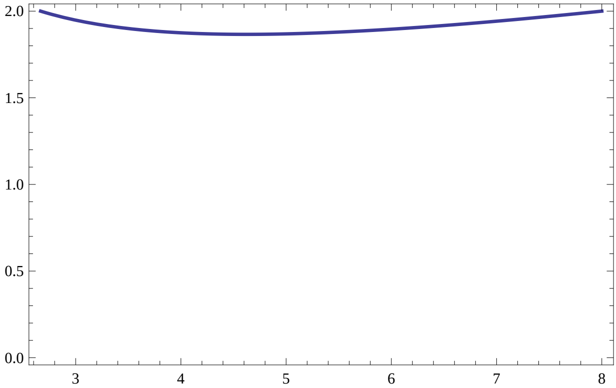
Gasket dimension

CLE_κ gasket dimension as a function of κ:



On an absolute scale:



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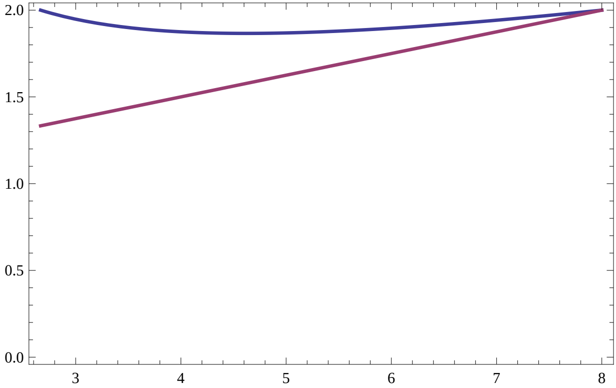

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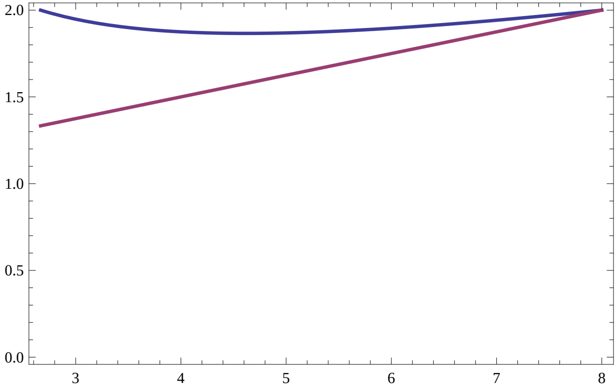


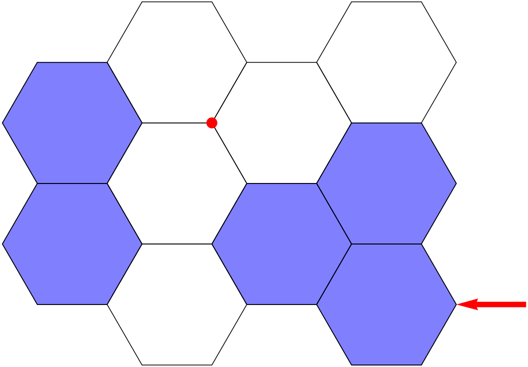
In fact, gasket is also closure of union of loops for $\kappa > 8/3$

Outline

1. The $O(n)$ model
2. Hausdorff dimension of the CLE gasket
3. Exploring a CLE
4. Ideas for the lower bound
5. An SLE estimate

Outline

Exploration path P_v towards v:



Discrete exploration process

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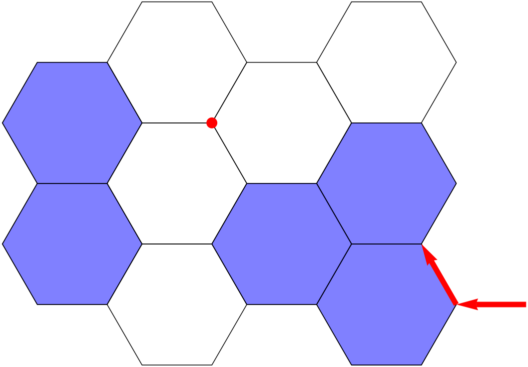

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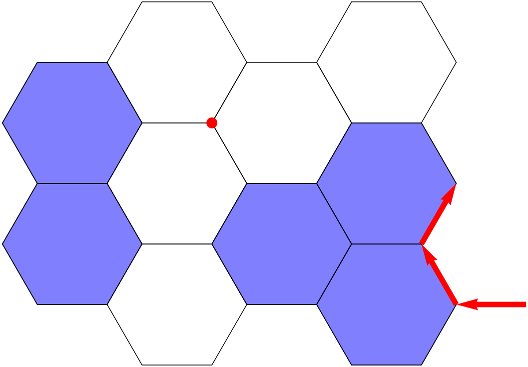

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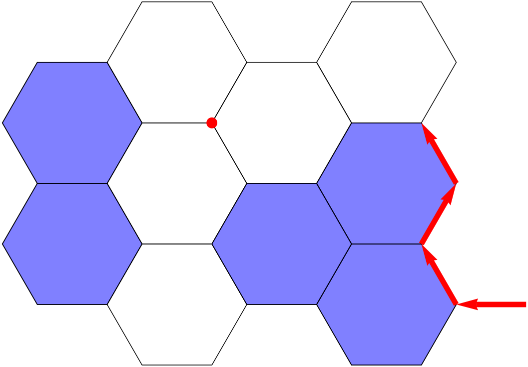

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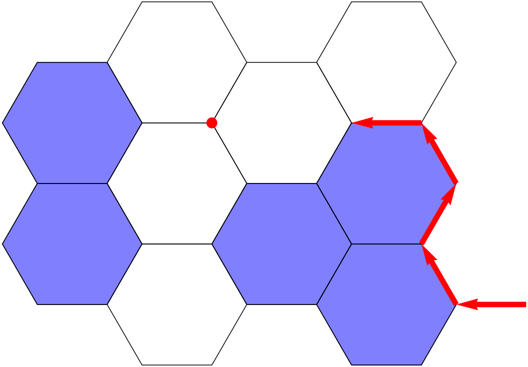

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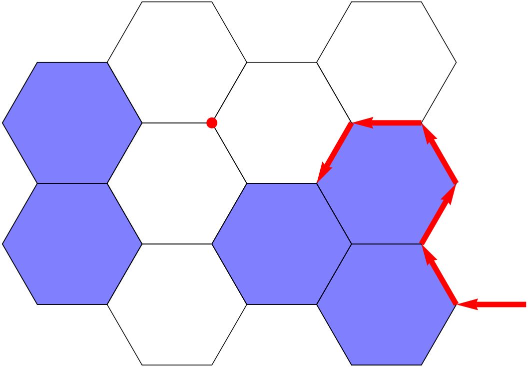

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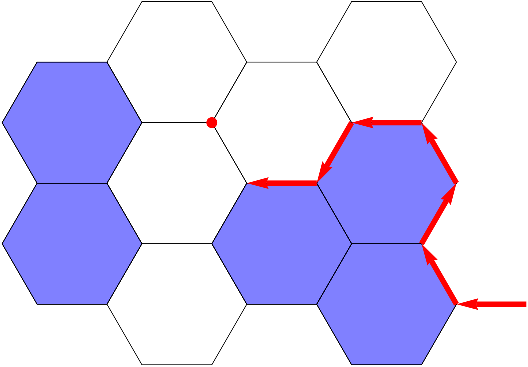

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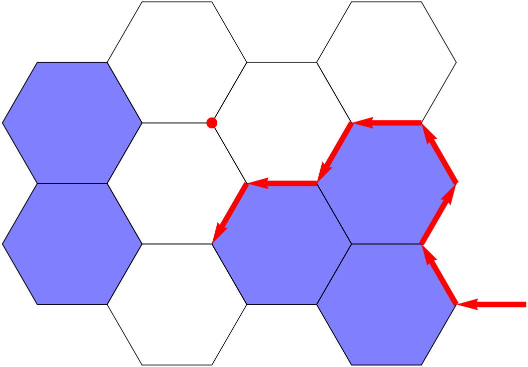

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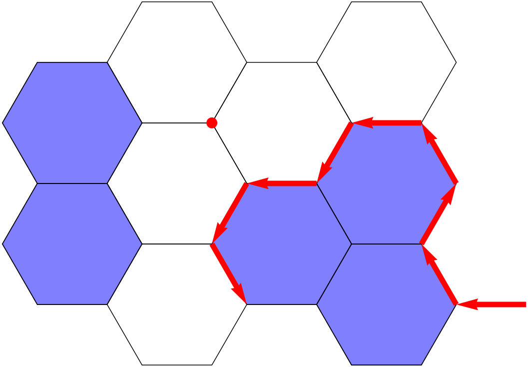

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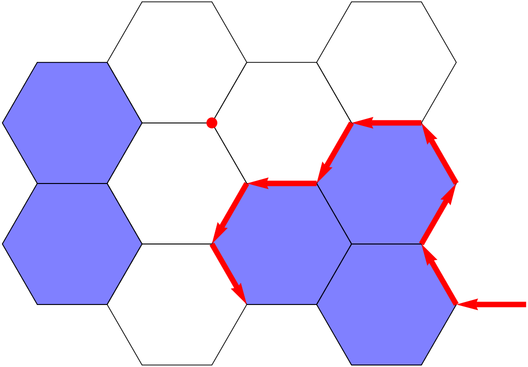

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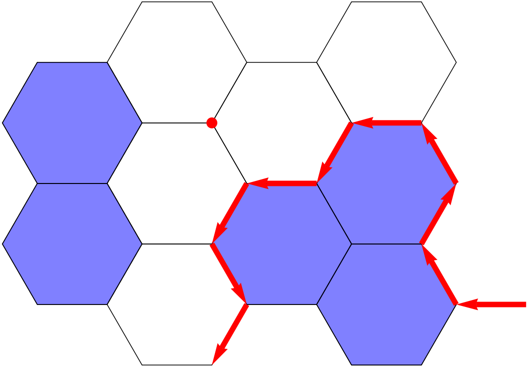

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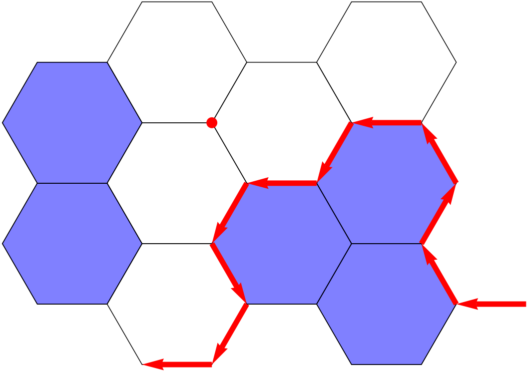

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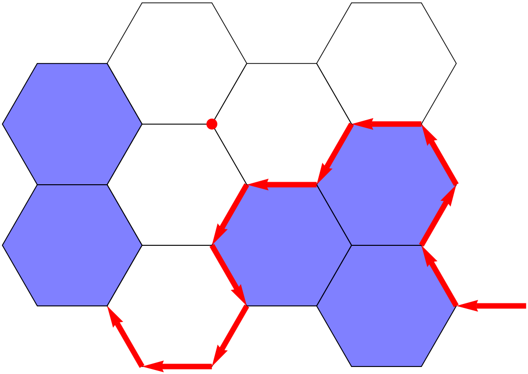

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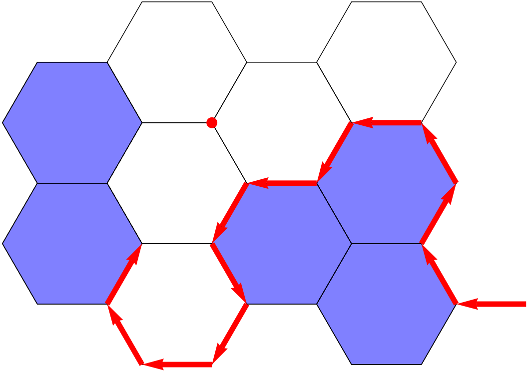

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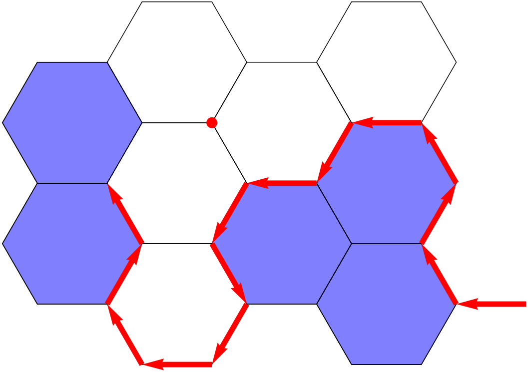

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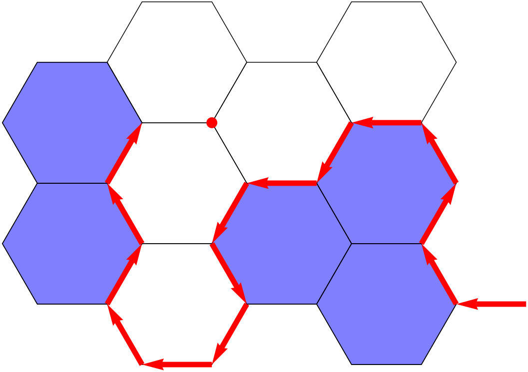

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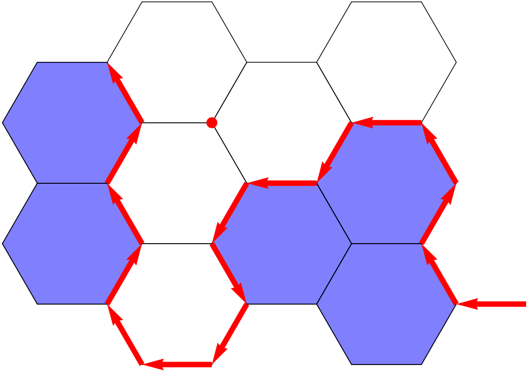

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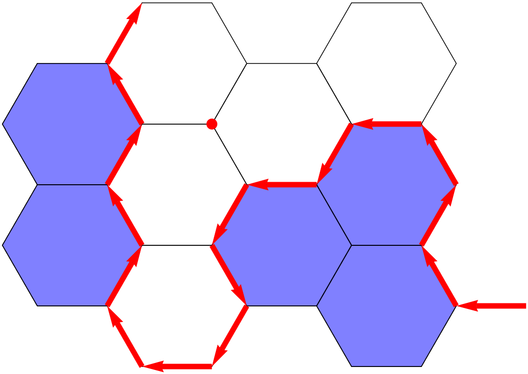

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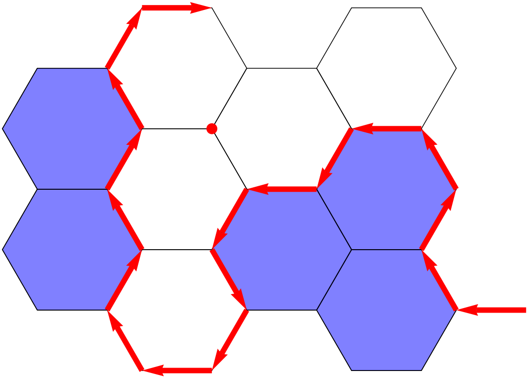

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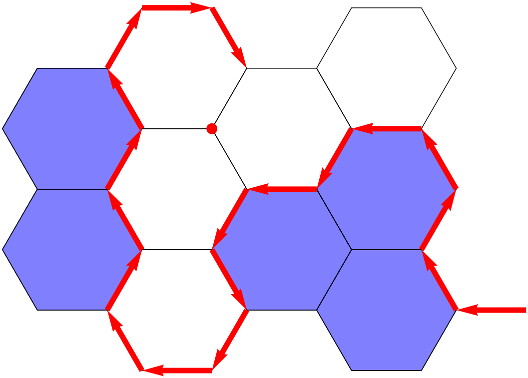

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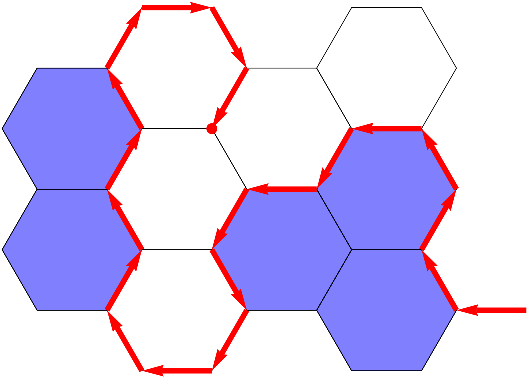

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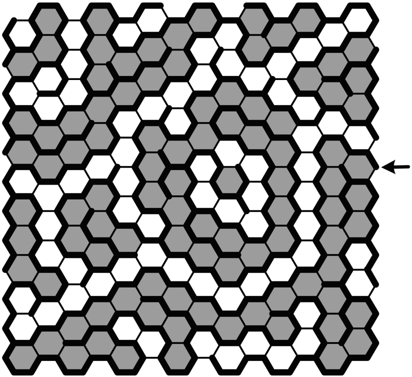

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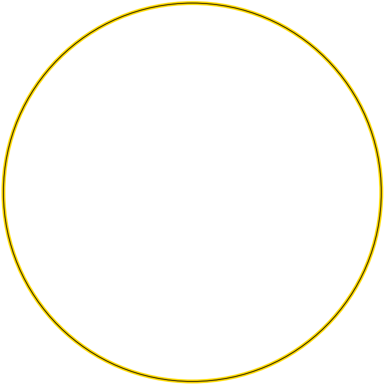
Union of all P_v is **exploration tree** [Sheffield Duke '09]

A branch of the exploration tree



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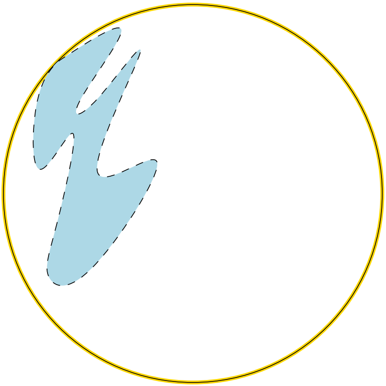

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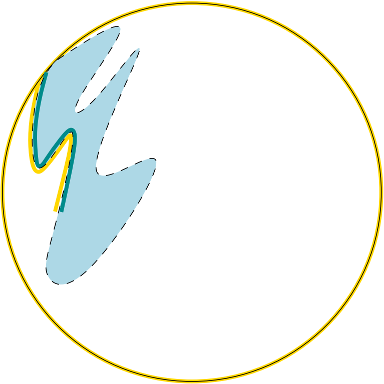

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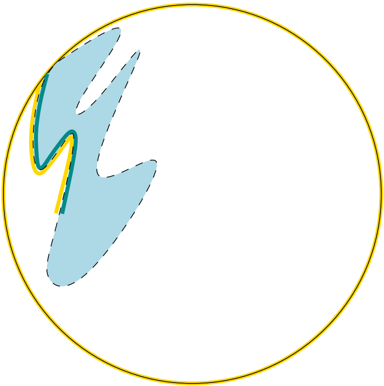

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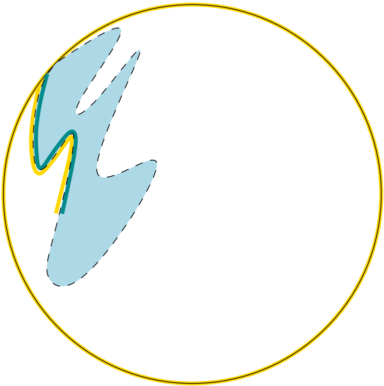

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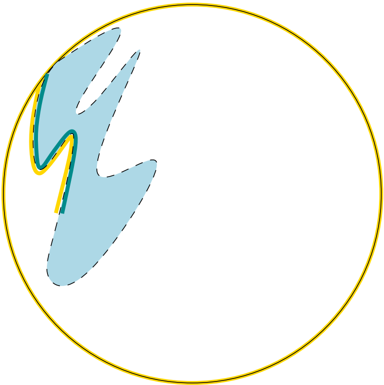

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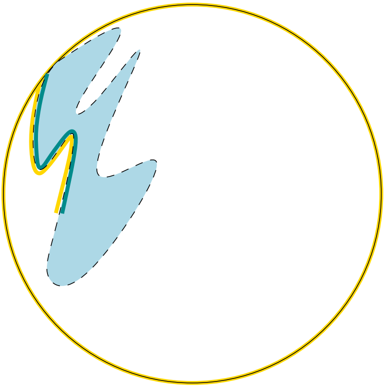

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Coordinate changes

A chordal Loewner evolution in $\mathbb{D}$ from $w \in \partial\mathbb{D}$ to $o \in \partial\mathbb{D}$ $(w \neq o)$

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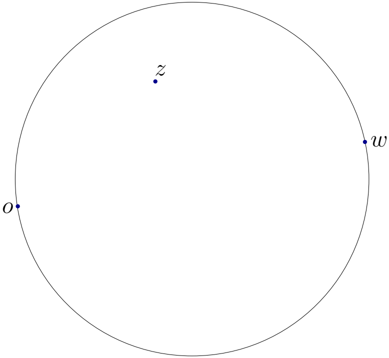

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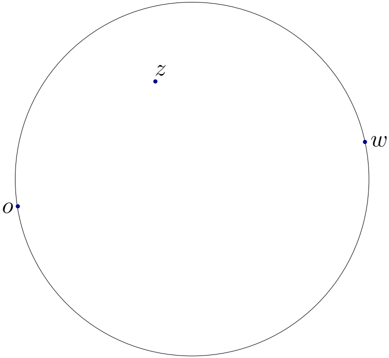

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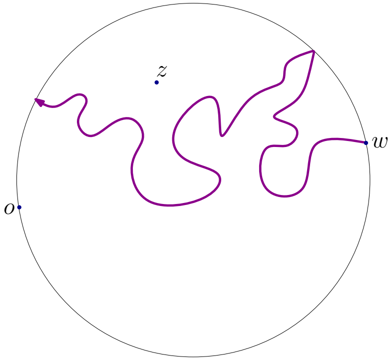

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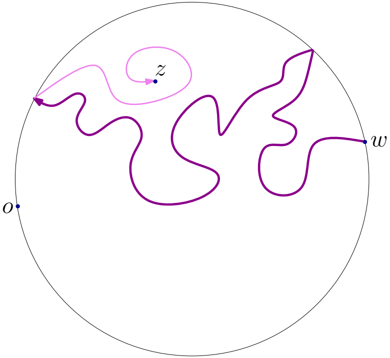

Coordinate changes

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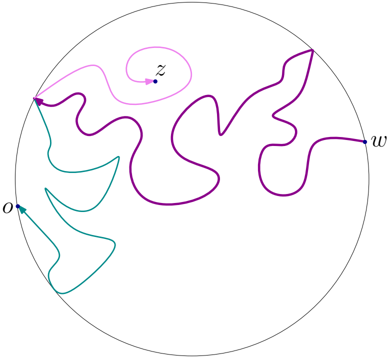

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[Schramm–Wilson NYJM '05]

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[Schramm–Wilson NYJM '05]

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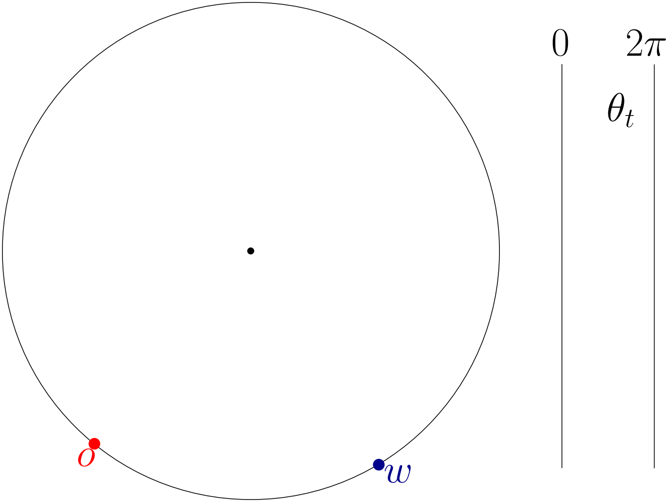
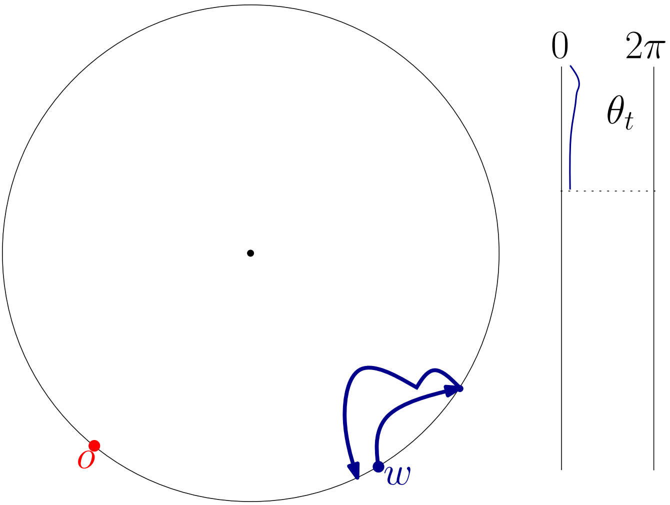
$= 2\pi$ times probability BM started from 0 hits between o and $\gamma(t)$

Radial $\mathrm{SLE}_\kappa(\kappa - 6)$




O
w
0
2π
θ_t

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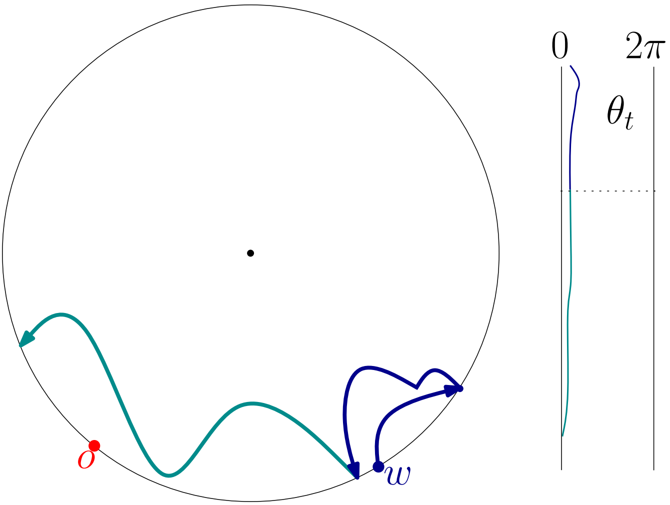

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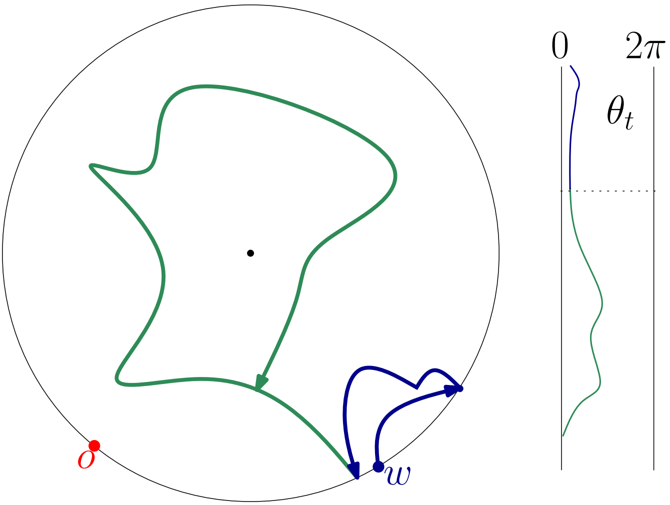

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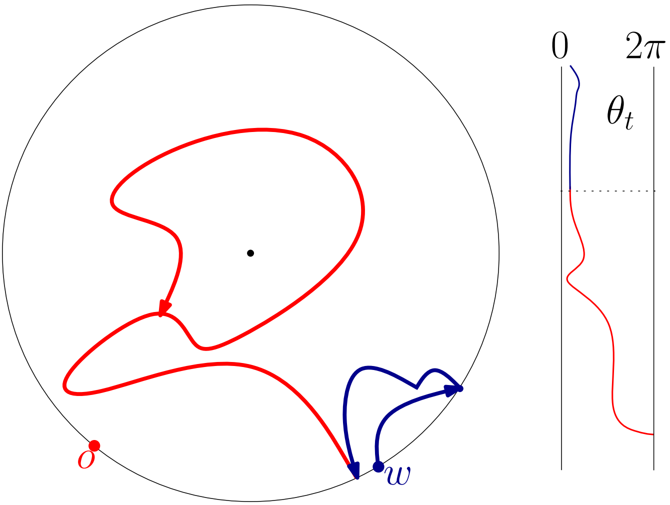

Chordal SLE_κ and radial $\mathrm{SLE}_\kappa(\kappa-6)$

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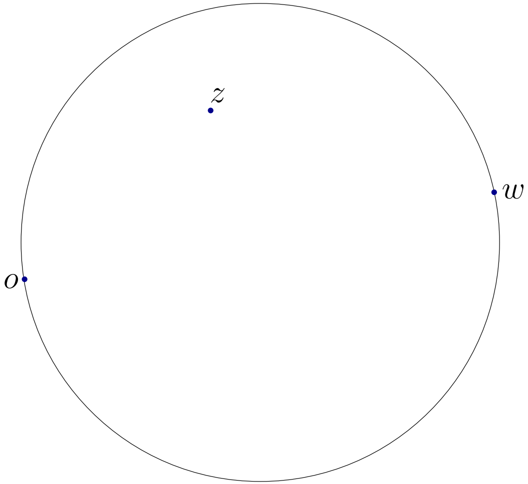

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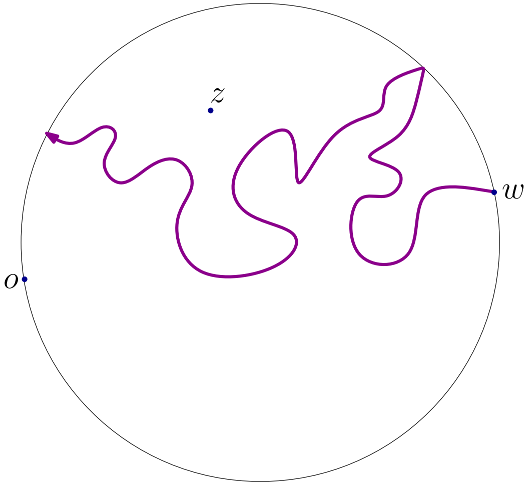

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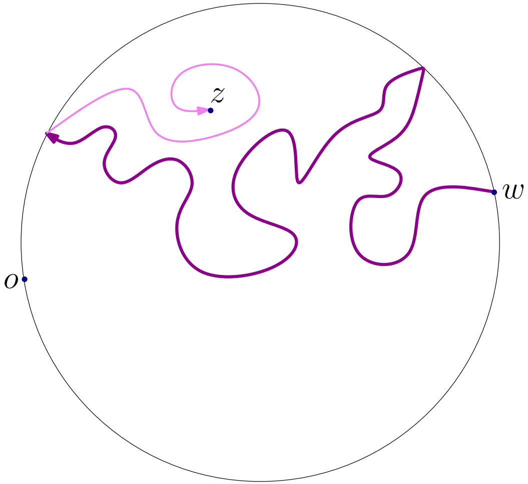

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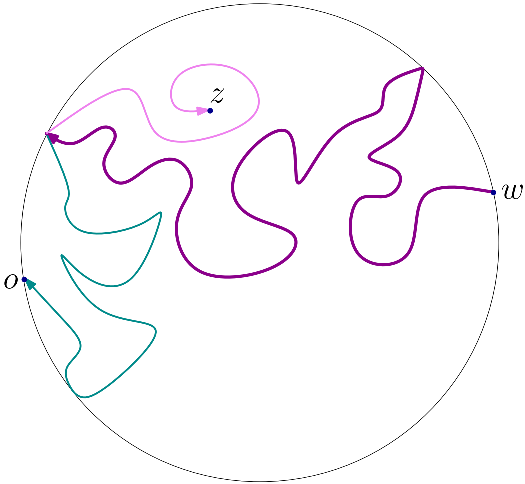

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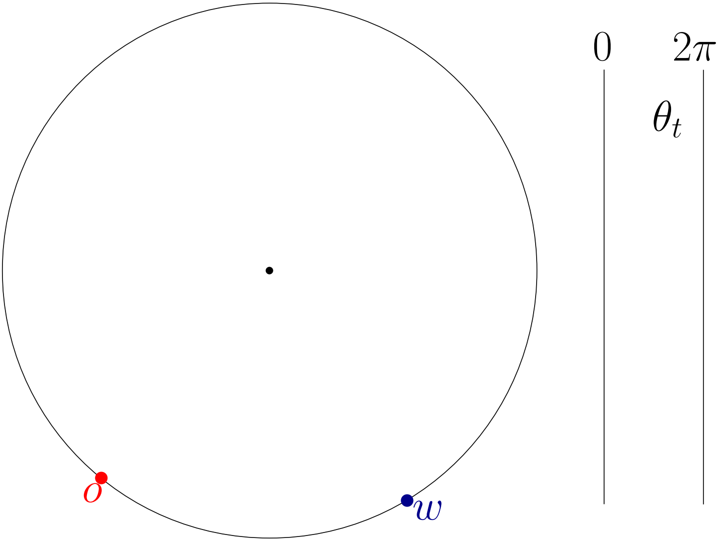

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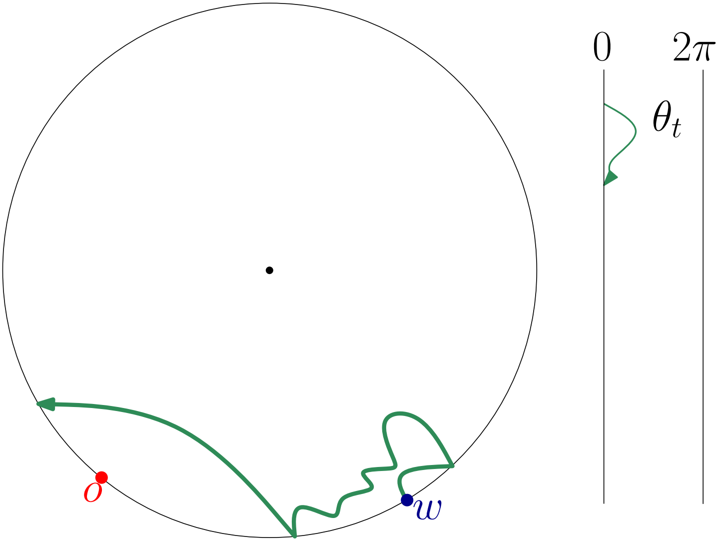

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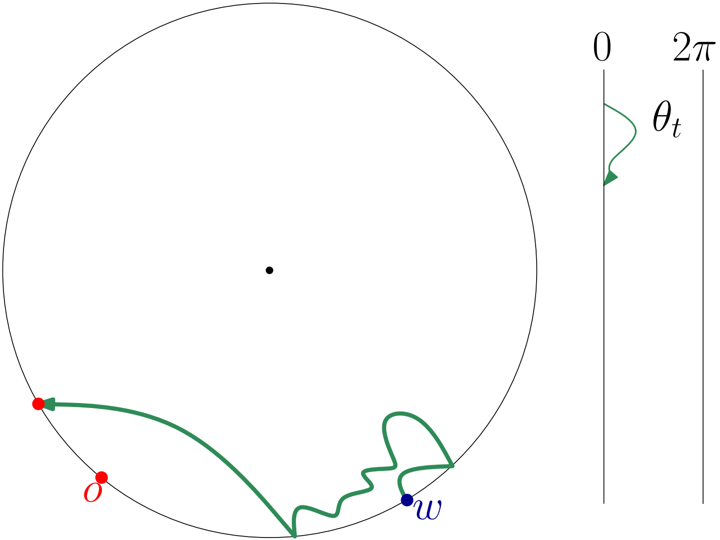

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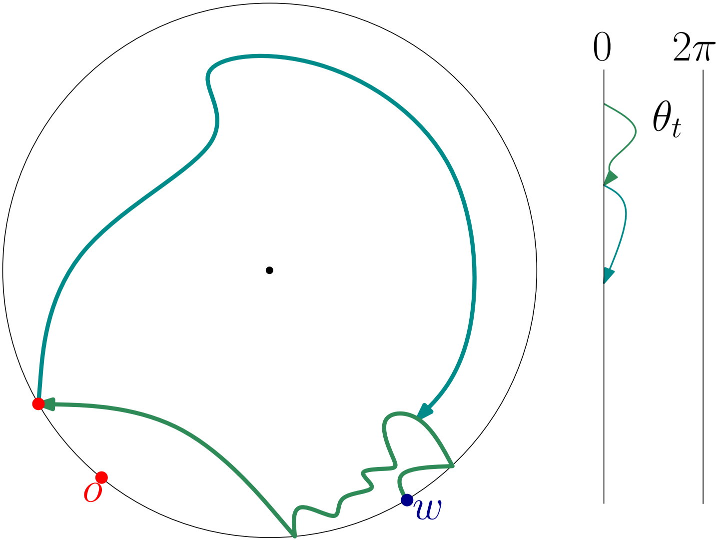

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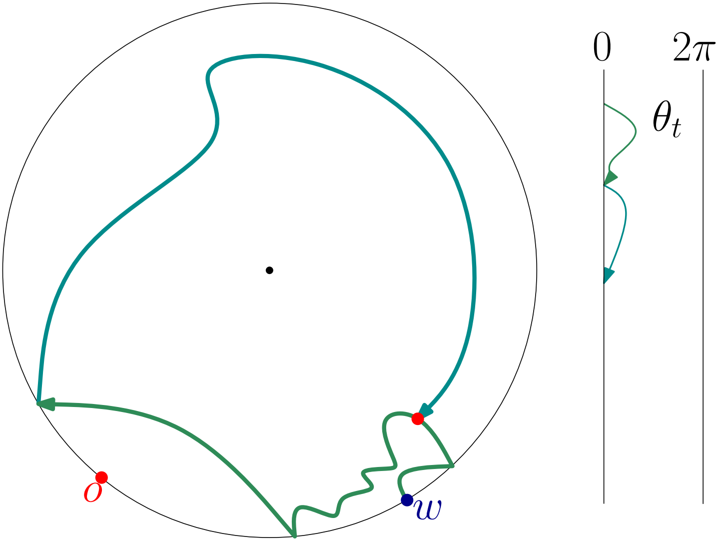

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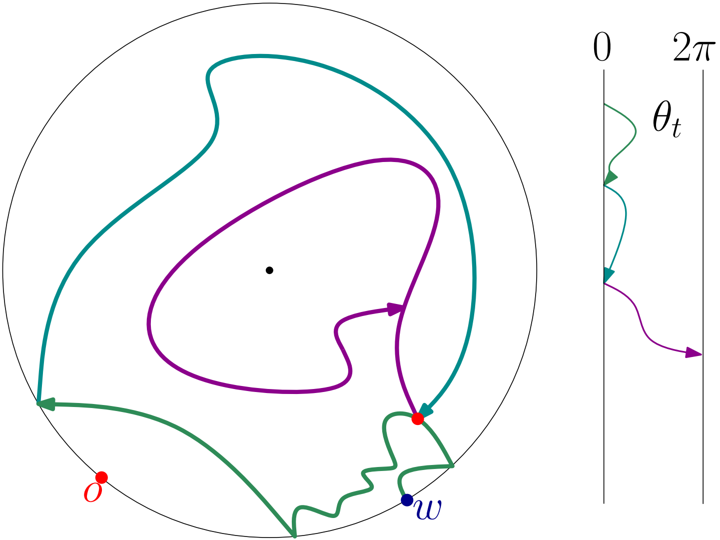

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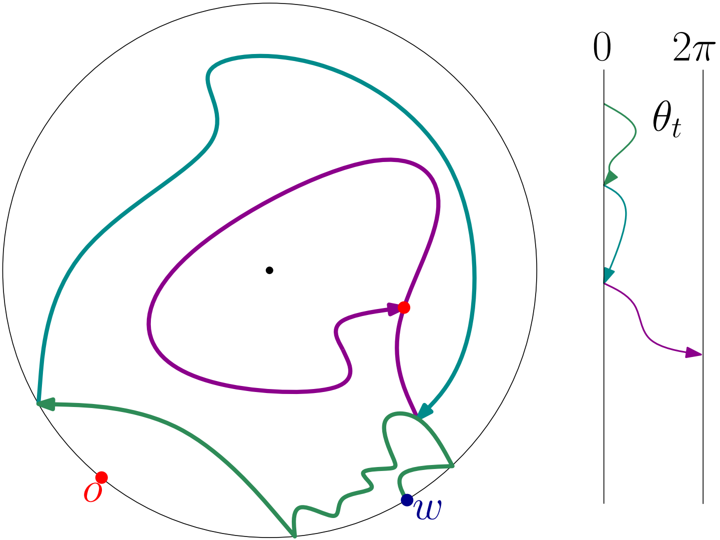

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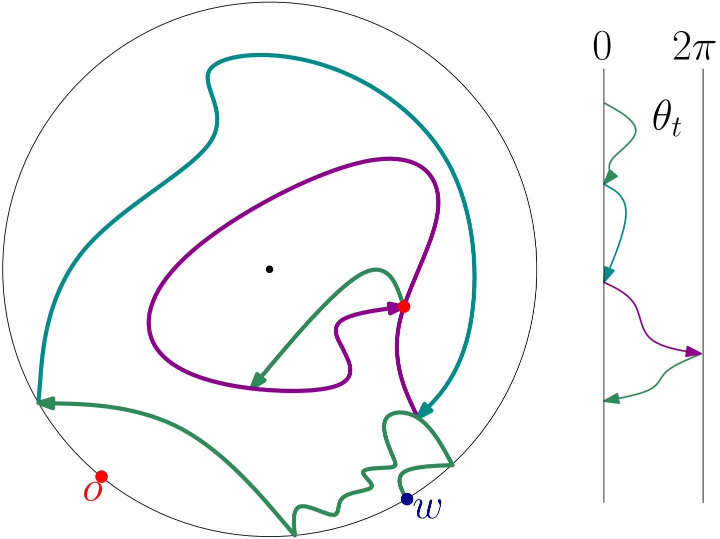

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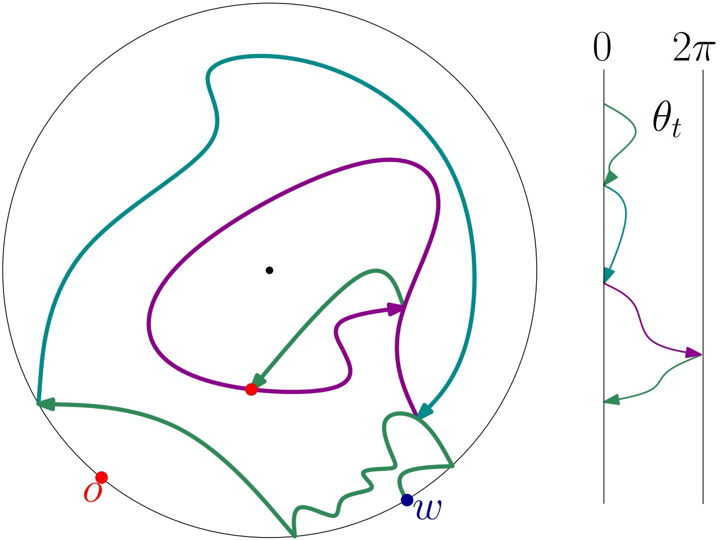

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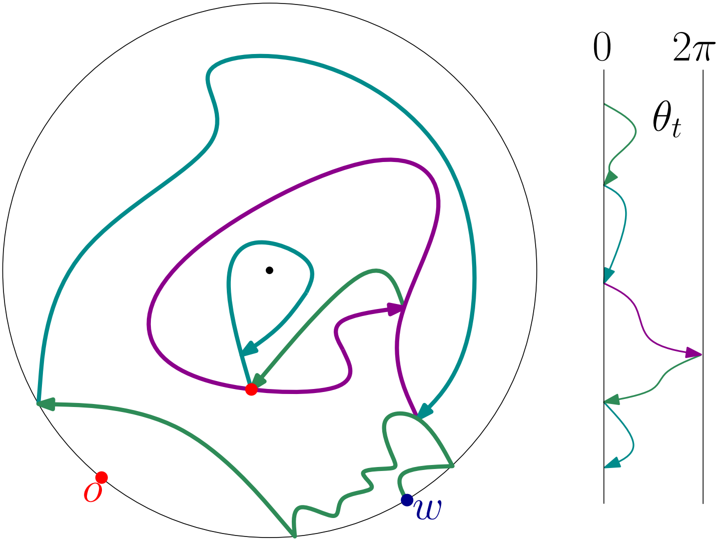

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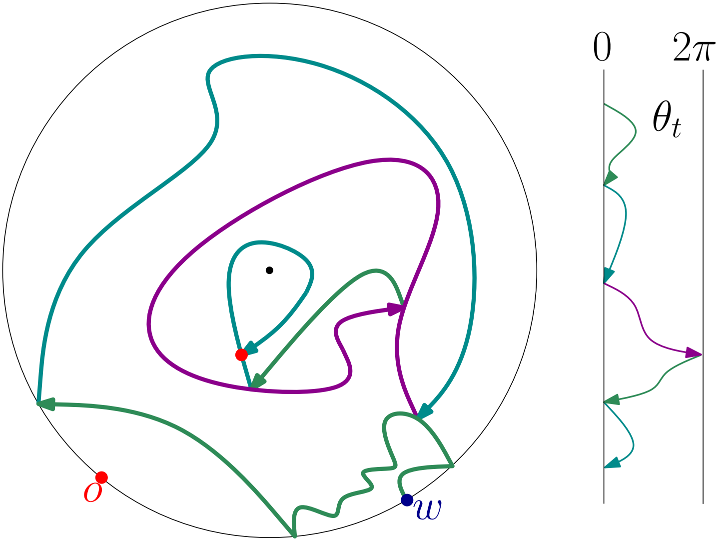

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branching SLE_6 [figure: Jason Miller]

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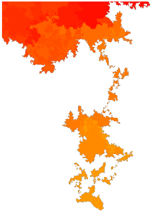

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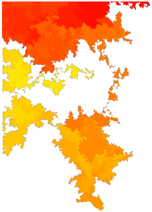

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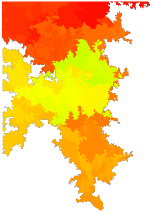

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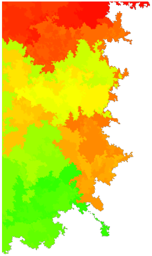

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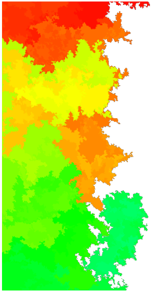

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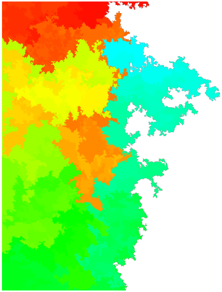

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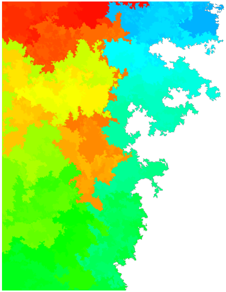

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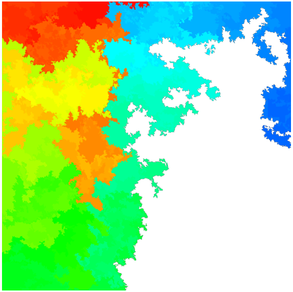

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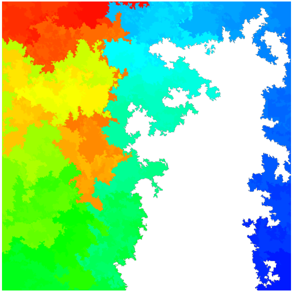

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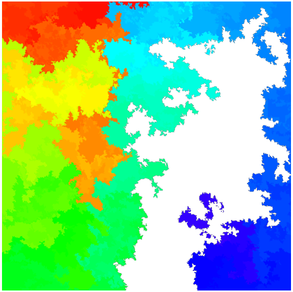

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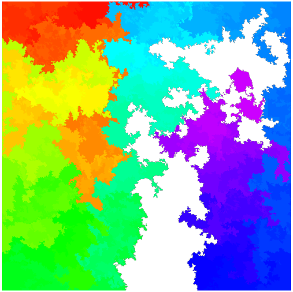

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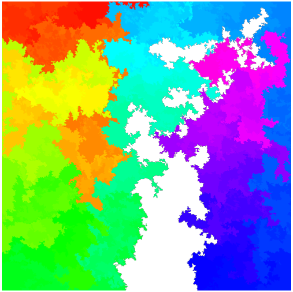

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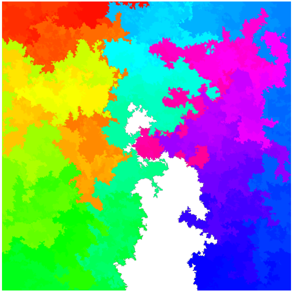

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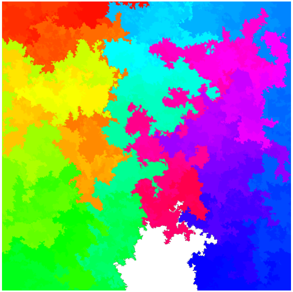

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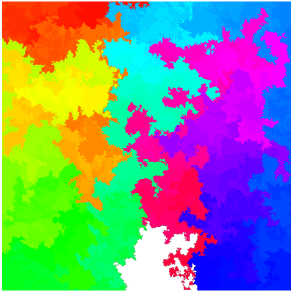

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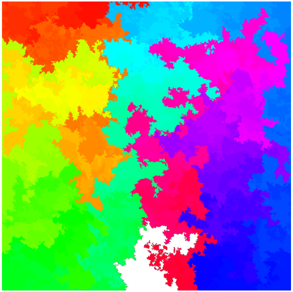

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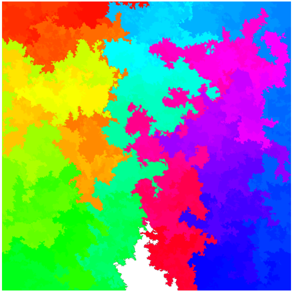

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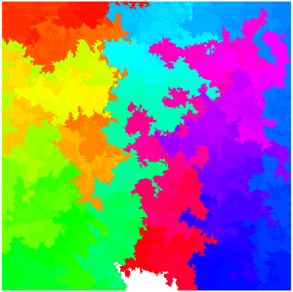

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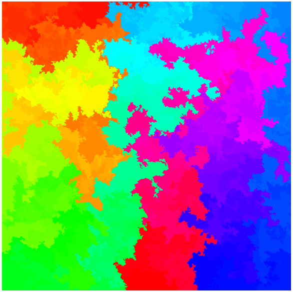

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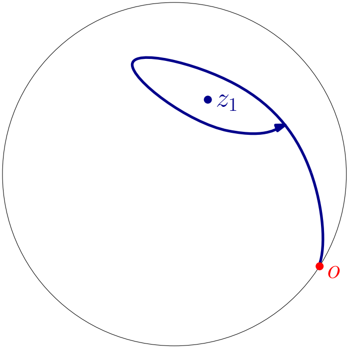

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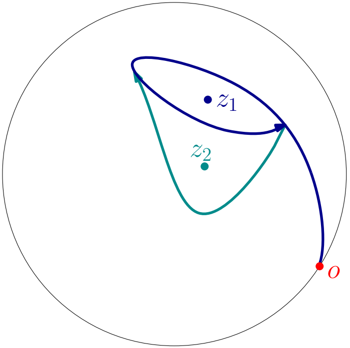

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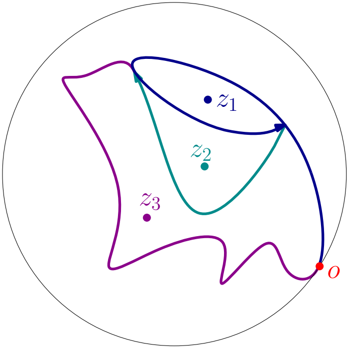

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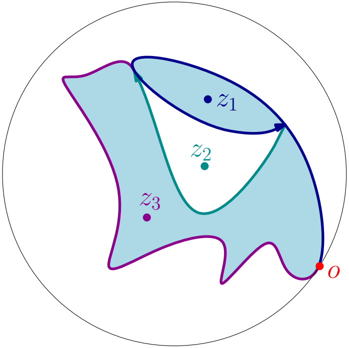

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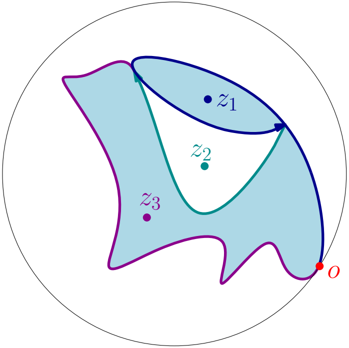

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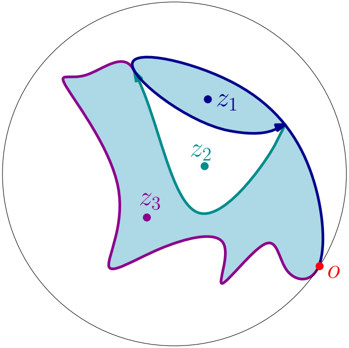

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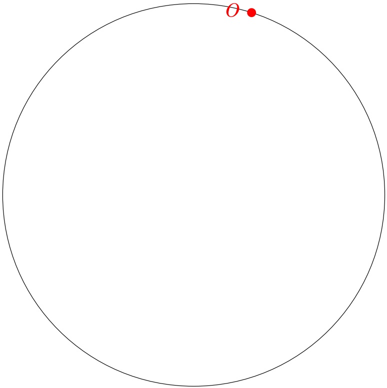

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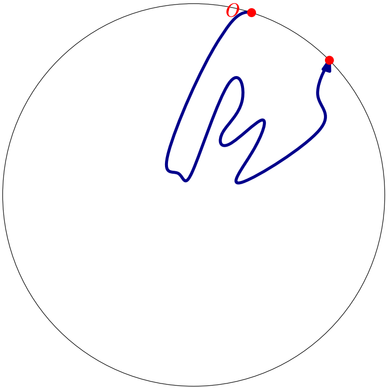

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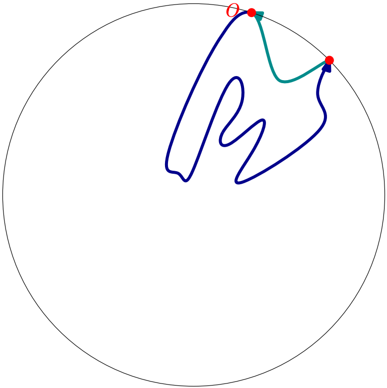

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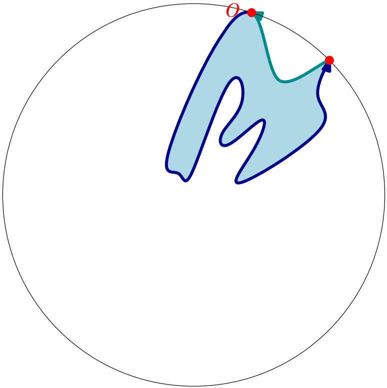

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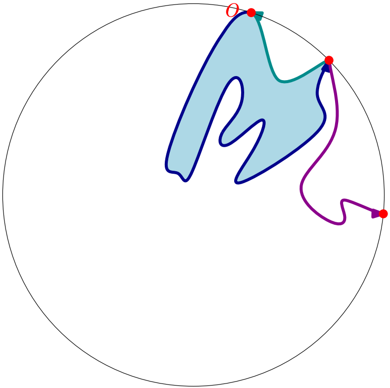

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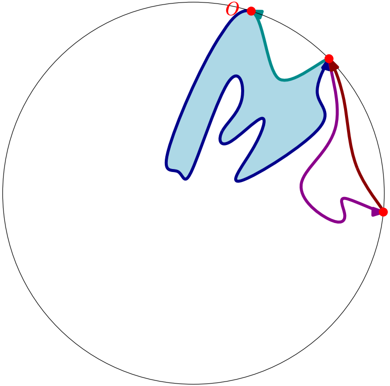

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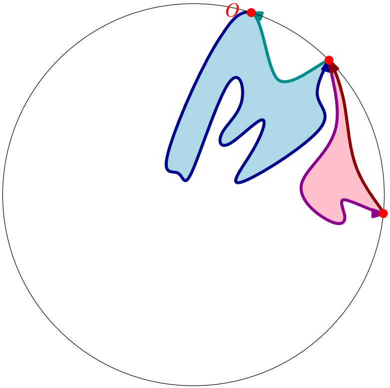

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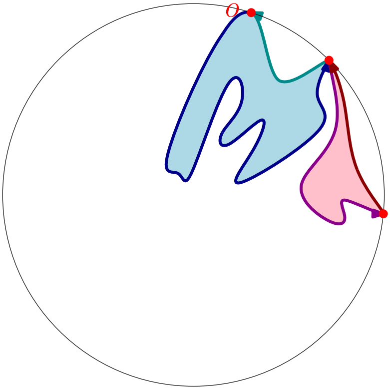

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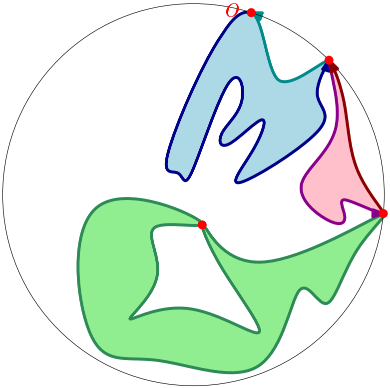

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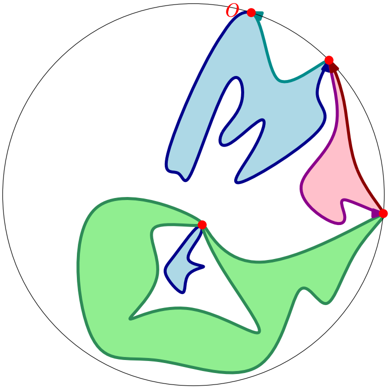

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Conjecture proved by works of Miller–Sheffield '12

Outline

1. The $O(n)$ model
2. Hausdorff dimension of the CLE gasket
3. Exploring a CLE
4. Ideas for the lower bound
5. An SLE estimate

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We prove the dimension lower bound on the CLE gasket by a multi-scale refinement of the second moment method originating from [Dembo–Peres–Rosen–Zeitouni AOP '00, Acta '01]

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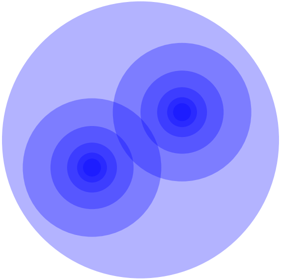

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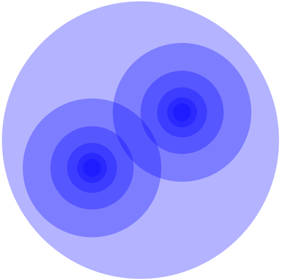

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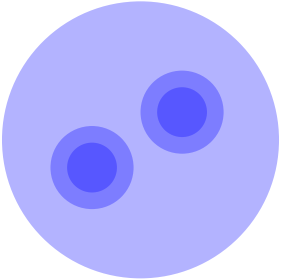

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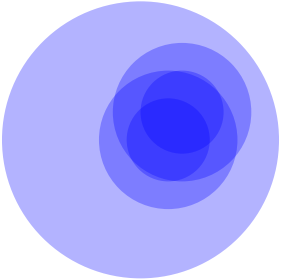

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rearranging and using $\mathbb{P}(E_{j+1}^z \,|\, E_j^z) \approx (e^{-\beta})^{s}$ gives $(\bigstar)$

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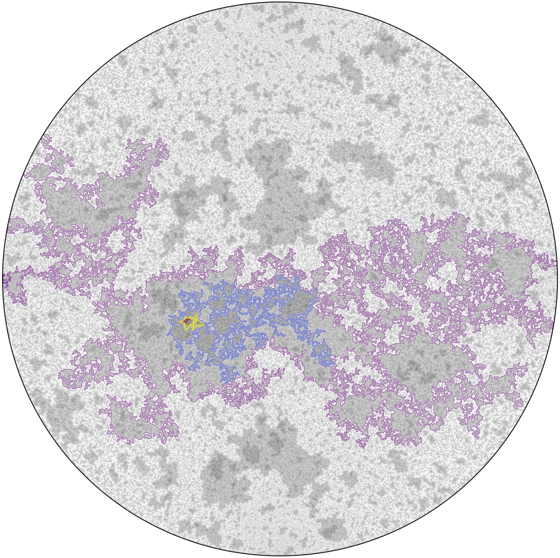
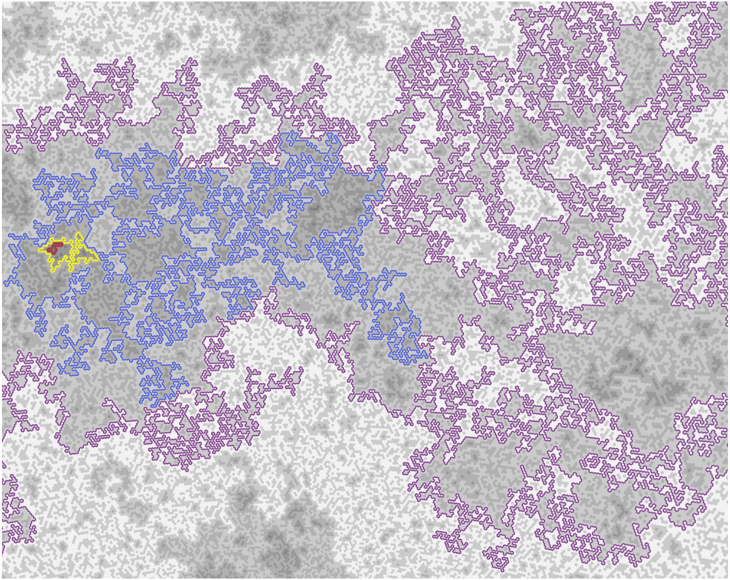
$\{z, w \in \Gamma^m\}$ may cause $\{z \in \Gamma^n\}, \{w \in \Gamma^n\}$ to be correlated



[figure: Sam Watson and David Wilson]



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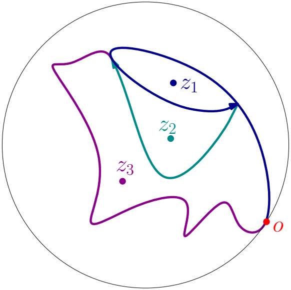

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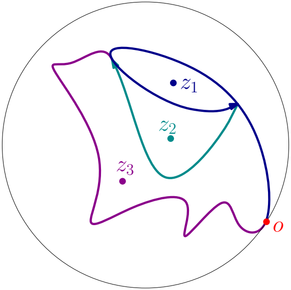
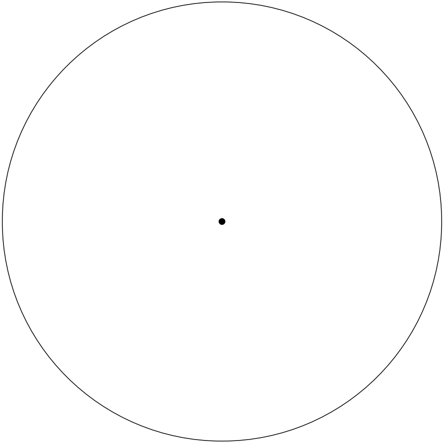


— so we use the CW loops to create the tree structure

Definition of E_1^0



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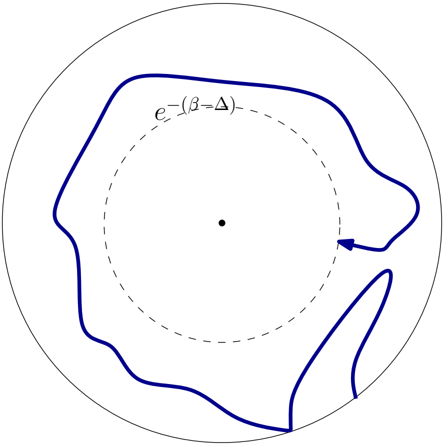

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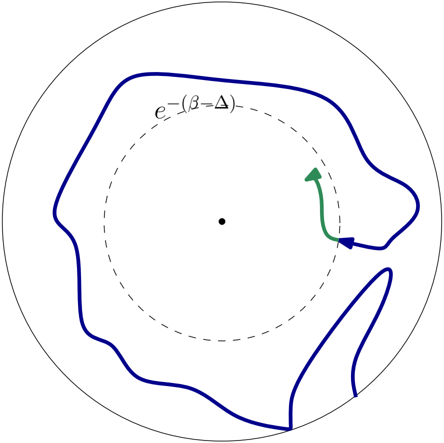

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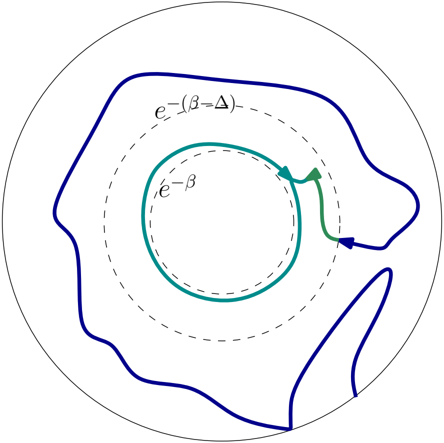

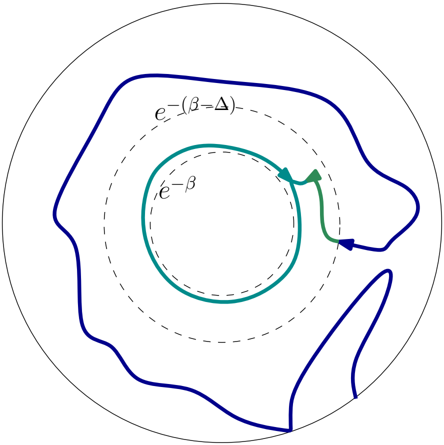

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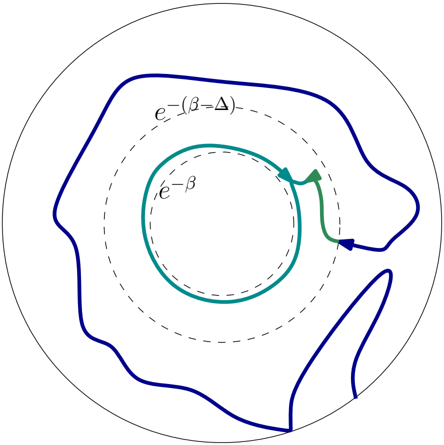

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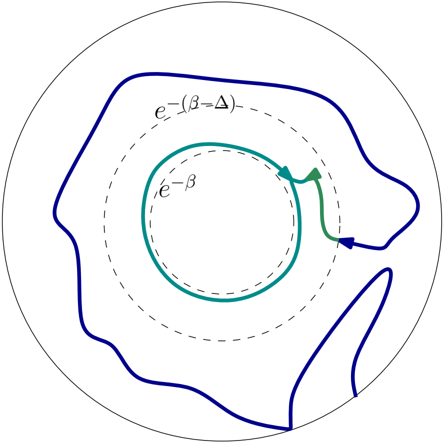

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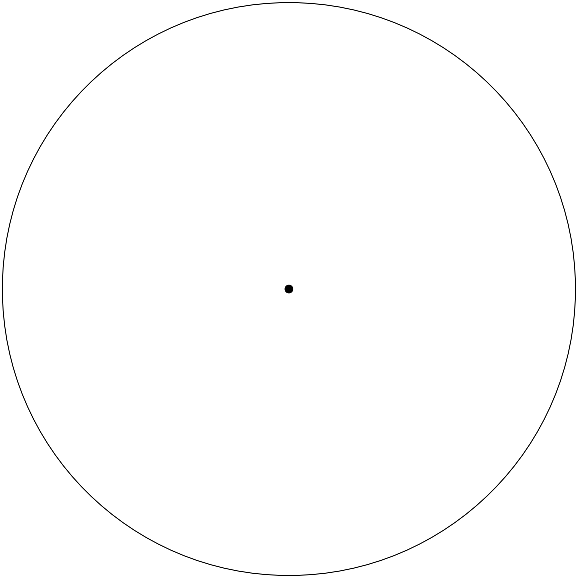
Remains to show $\mathbb{P}(E_1^0) \approx (e^{-\beta})^{\alpha}$

Outline

1. The $O(n)$ model
2. Hausdorff dimension of the CLE gasket
3. Exploring a CLE
4. Ideas for the lower bound
5. An SLE estimate

Outline

Probability of E_1^0



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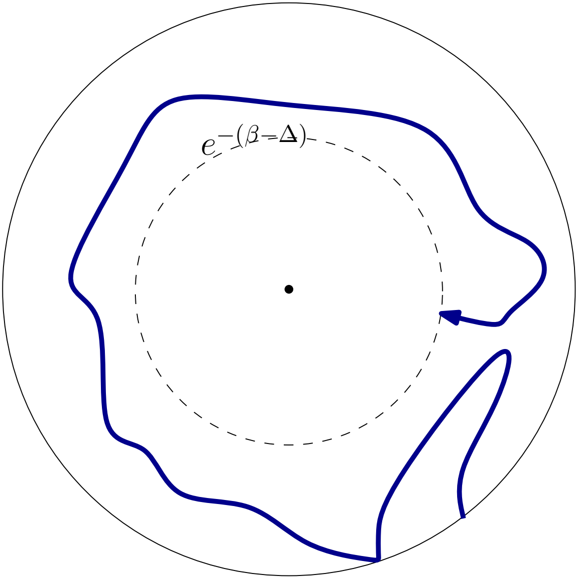

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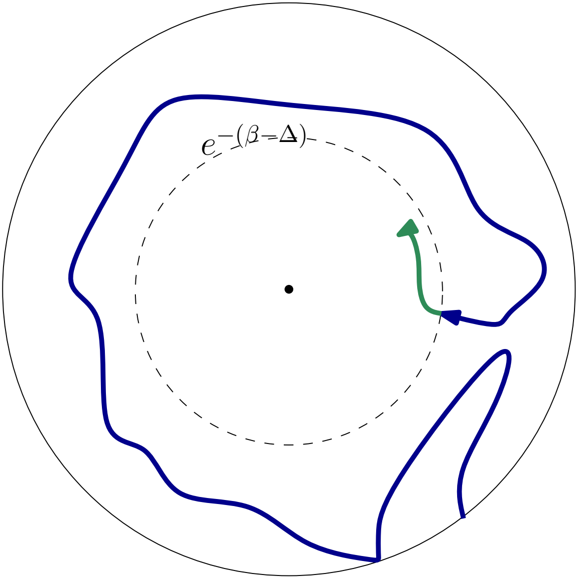

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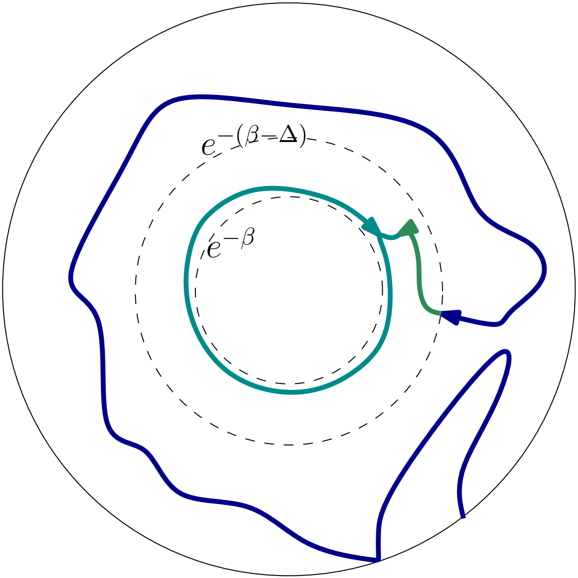

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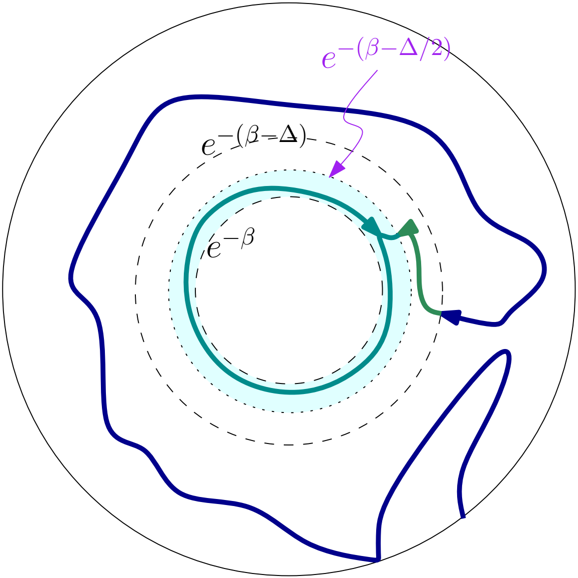

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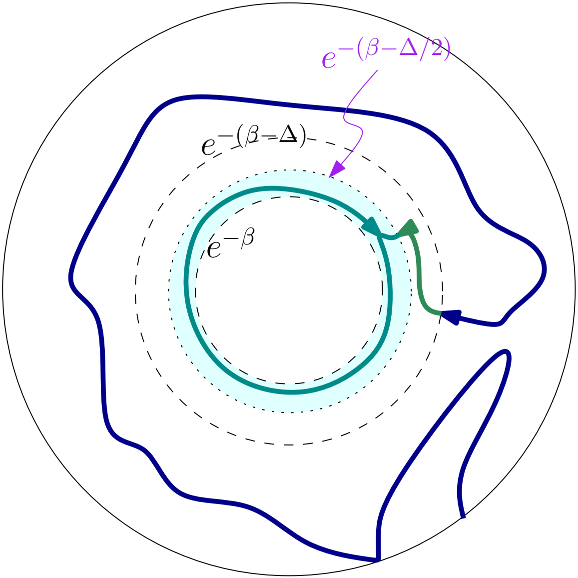
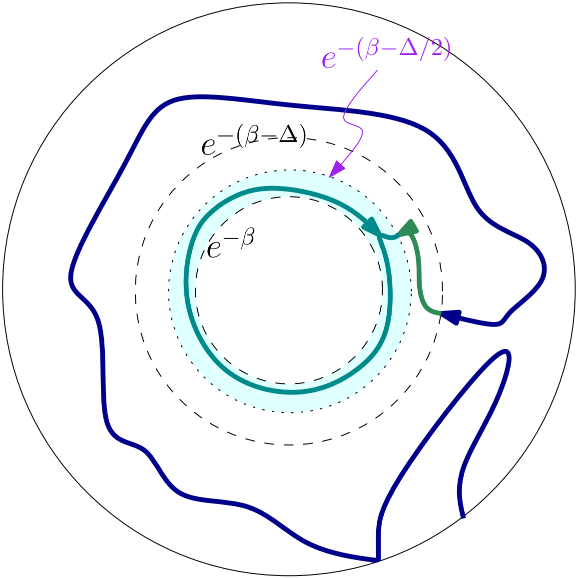


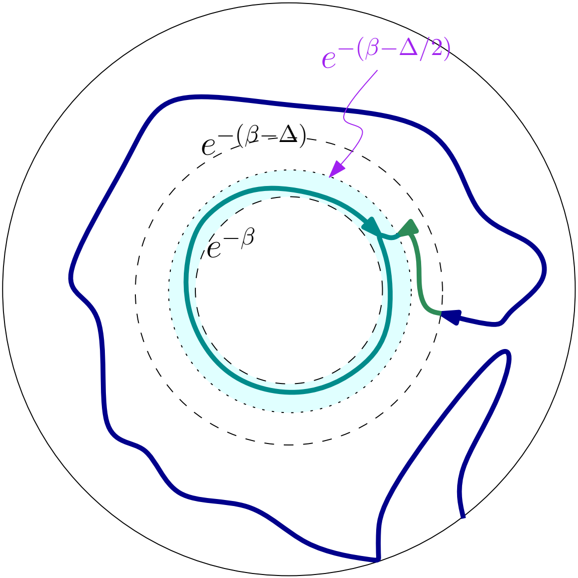
Stage 1:

Probability of E_1^0



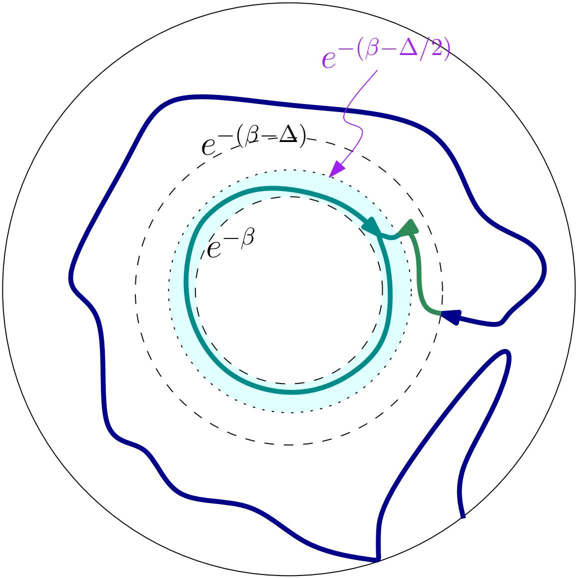
Stage 1:
reach $e^{-(\beta-\Delta)}$

Probability of E_1^0



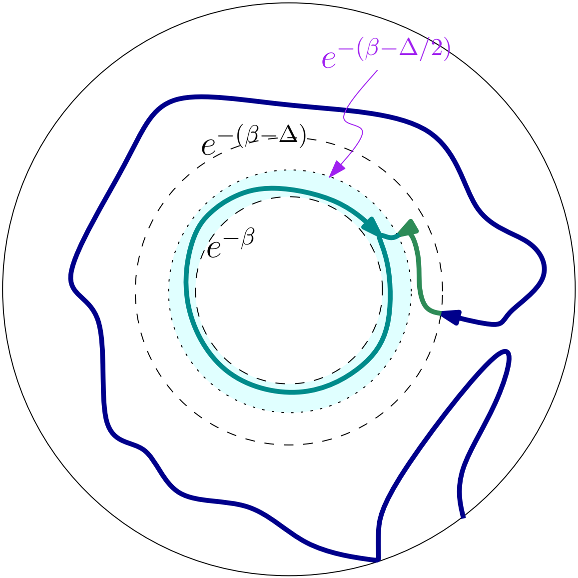
Stage 1:
reach $e^{-(\beta-\Delta)}$
without CCW loop
around 0:

Probability of E_1^0



Stage 1:
reach $e^{-(\beta-\Delta)}$
without CCW loop
around 0:
prob. $\asymp (e^{-\beta})^{\alpha}$
[SSW]

Probability of E_1^0



Stage 1:
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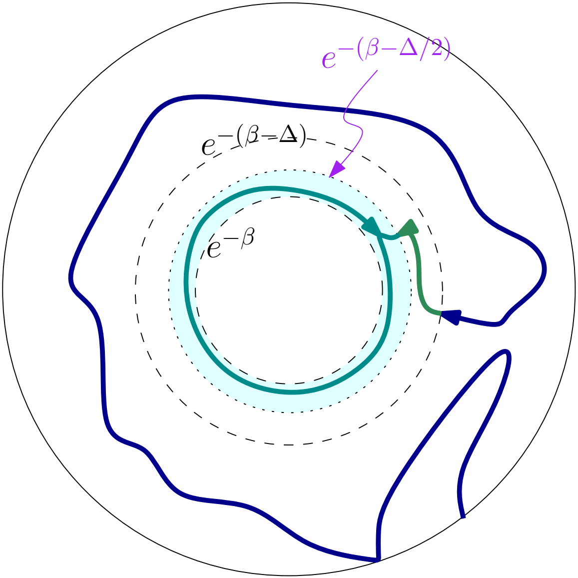
[SSW]

Stage 2:

Probability of E_1^0



Stage 1:
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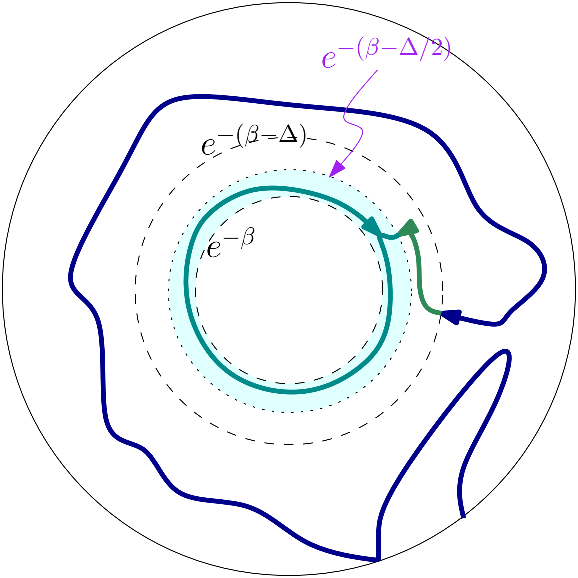
[SSW]

Stage 2:
CW loop within
inner annulus before
CCW loop around 0:

Probability of E_1^0



Stage 1:
reach $e^{-(\beta-\Delta)}$
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[SSW]

Stage 2:
CW loop within
inner annulus before
CCW loop around 0:
need to show
prob. $\asymp 1$

E_1^0 continued on Stage 1

Conditioned on **Stage 1** success at time τ_1, uniformize by g_{τ_1}:

E_1^0 continued on Stage 1

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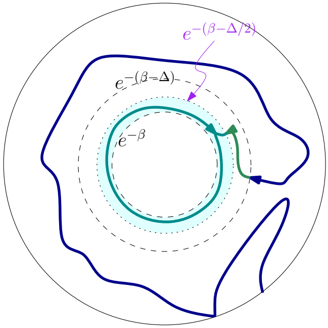

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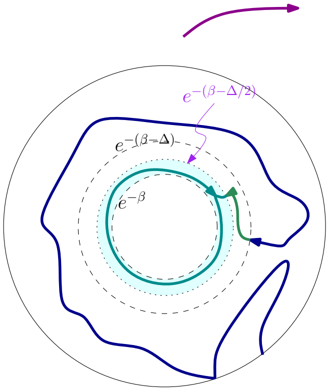

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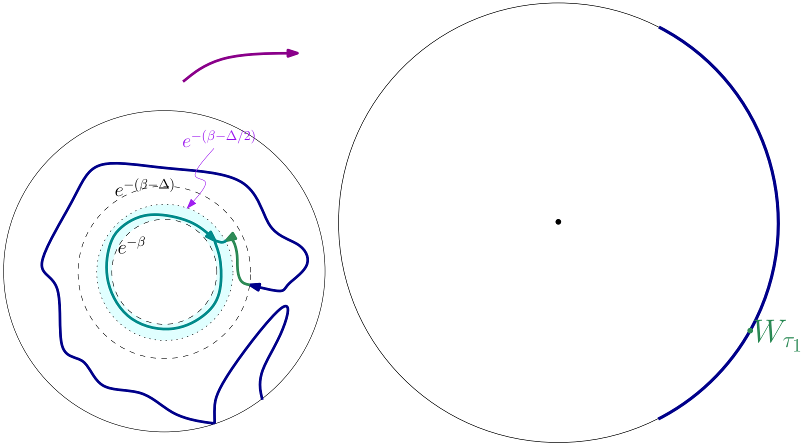

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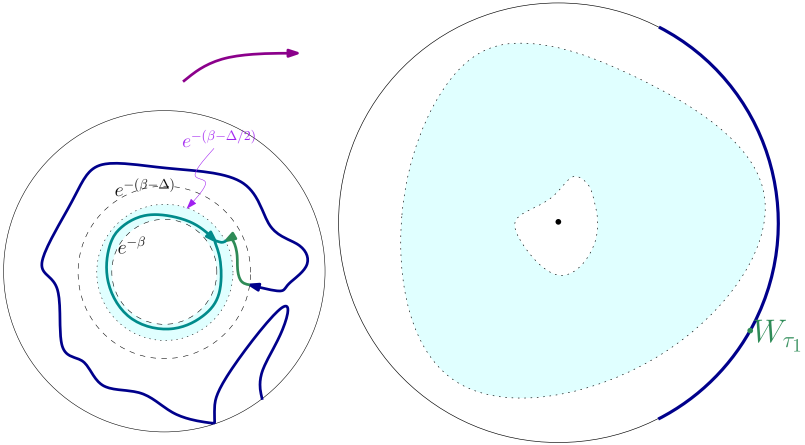

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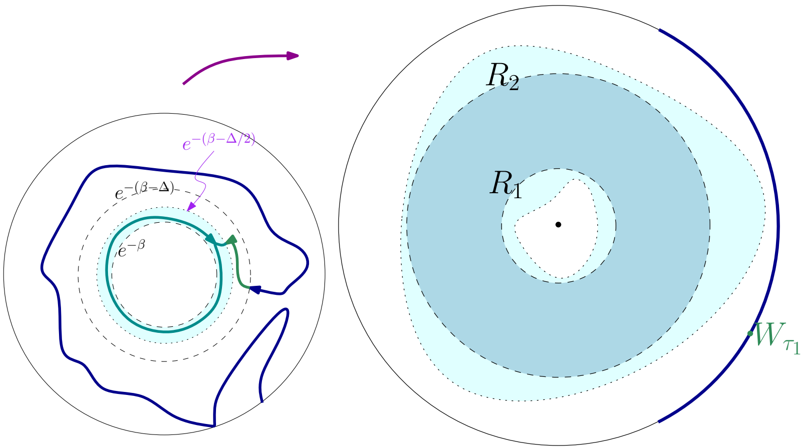

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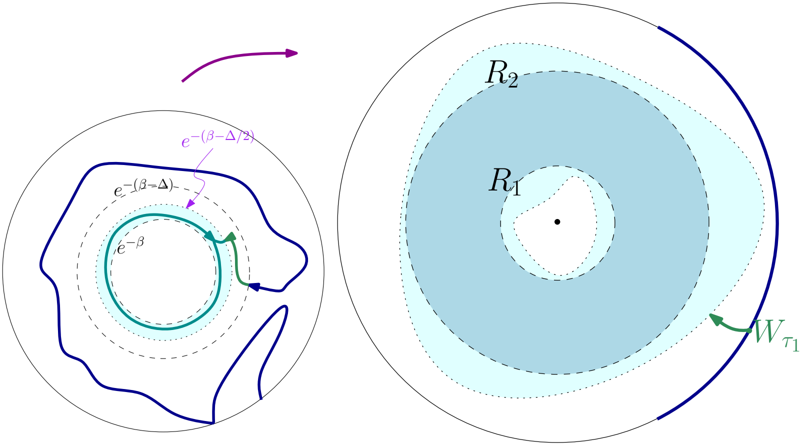

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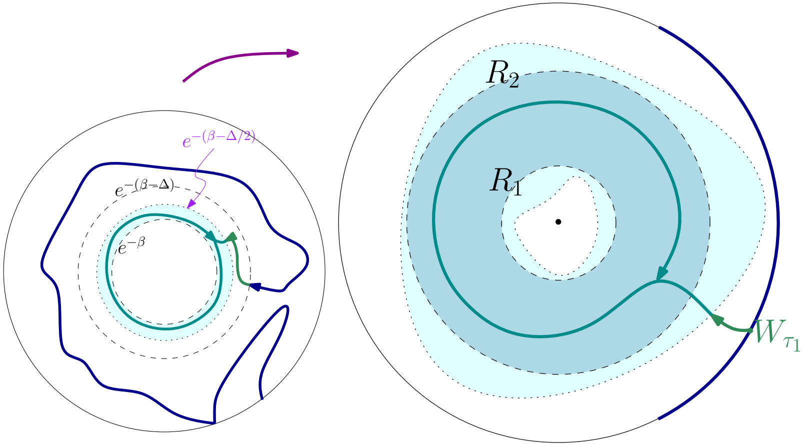

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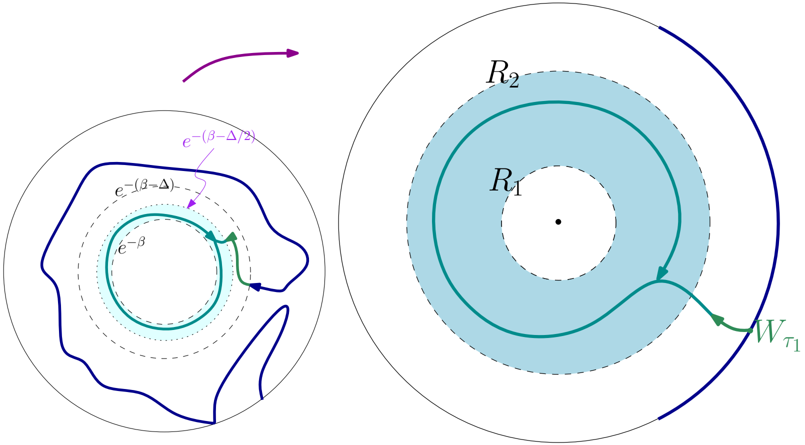

E_1^0 continued on Stage 1

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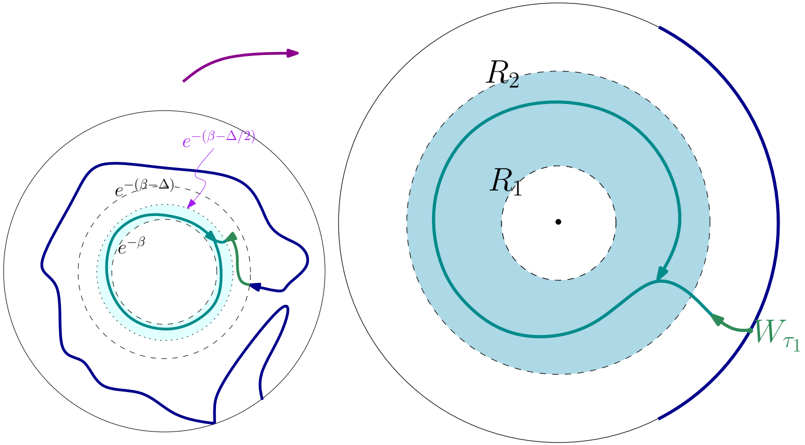

New **Stage 2**: CW loop in $A(0, R_1, R_2)$ before CCW loop surrounding 0

E_1^0 continued on Stage 1

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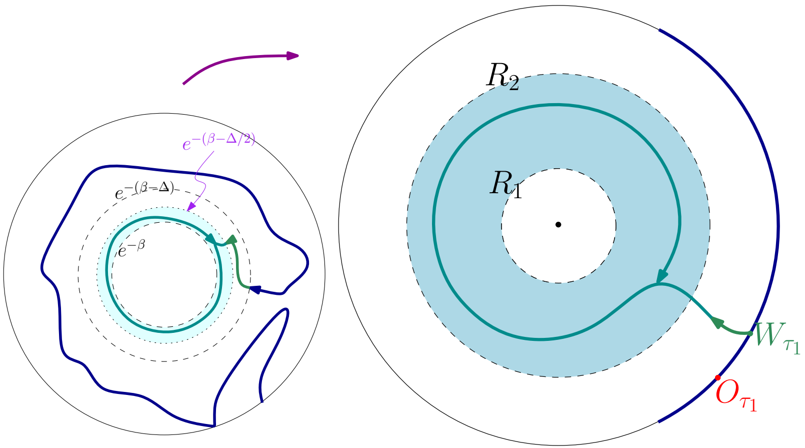

New **Stage 2**: CW loop in $A(0, R_1, R_2)$ before CCW loop surrounding 0

An almost loop

Chordal SLE_κ curve γ traveling $W_{\tau_1} \rightsquigarrow O_{\tau_1}$ makes almost-loop with positive probability:

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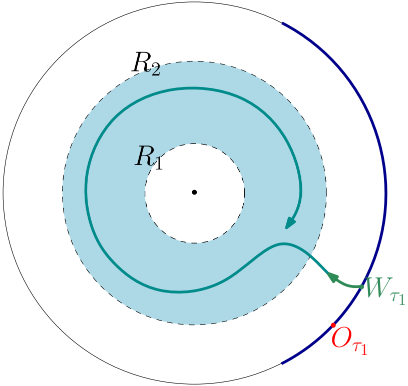

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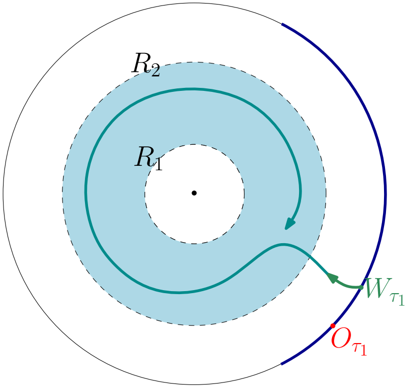

Driving functions uniformly close

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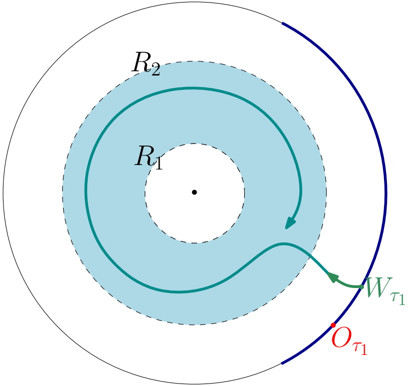

Driving functions uniformly close $\Rightarrow$ curves close in Carathéodory topology

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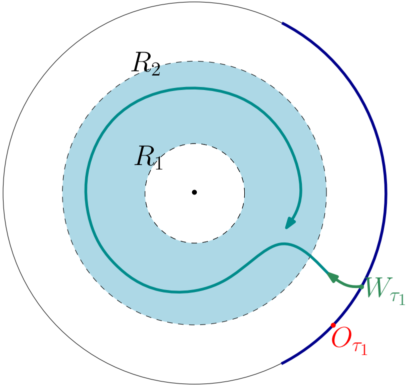

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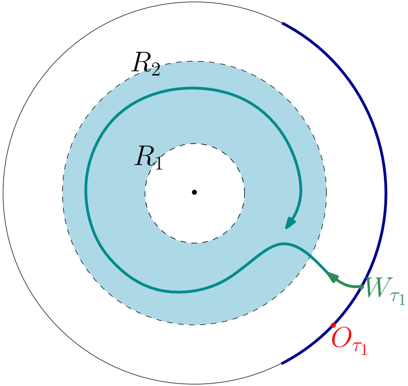

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Suffices for driving function to be within ϵ of fixed driving function

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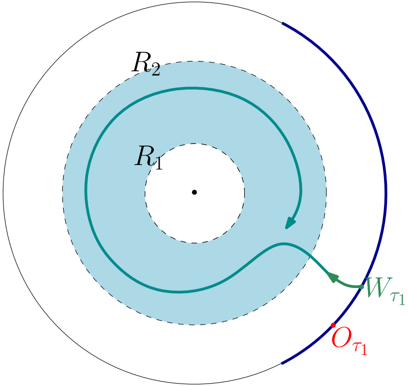

Driving functions uniformly close
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Suffices for driving function to be within ϵ of fixed driving function with ϵ uniform over all $|W - O| \geq c$

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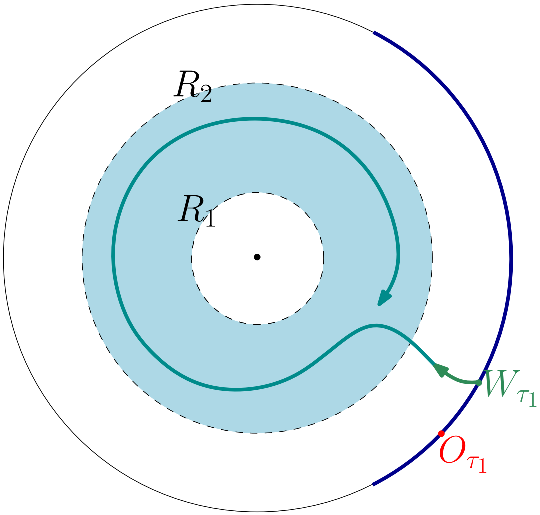
Combining parts (a) and (b) proves the claim

Closing the loop



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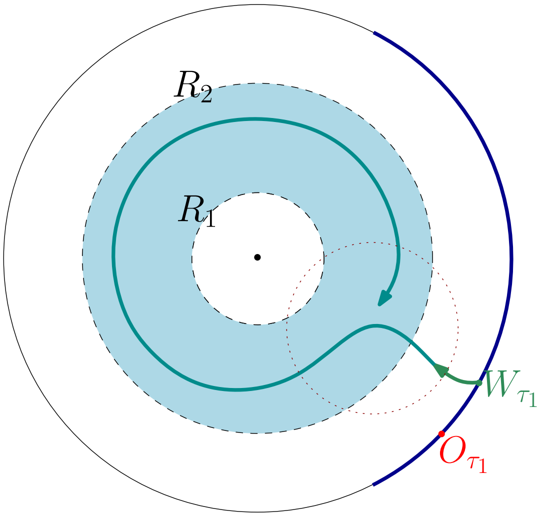

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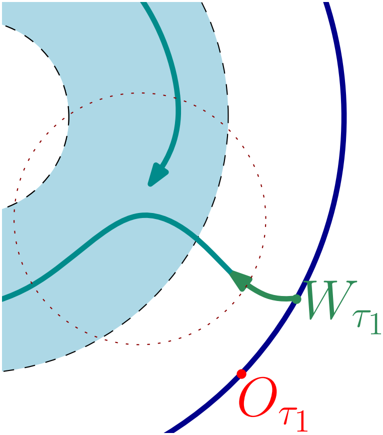

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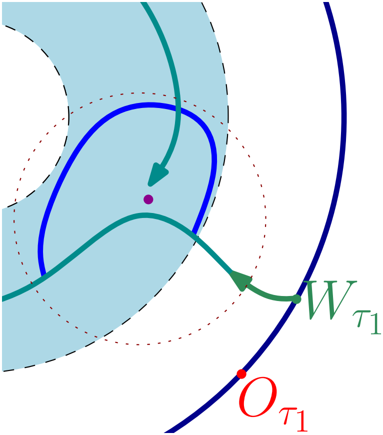

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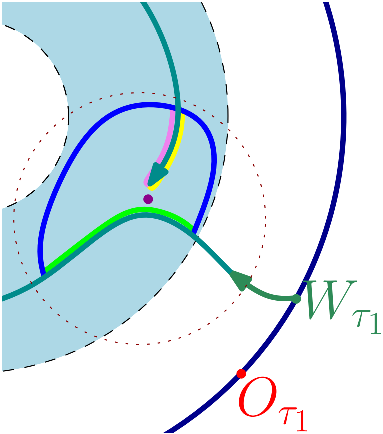

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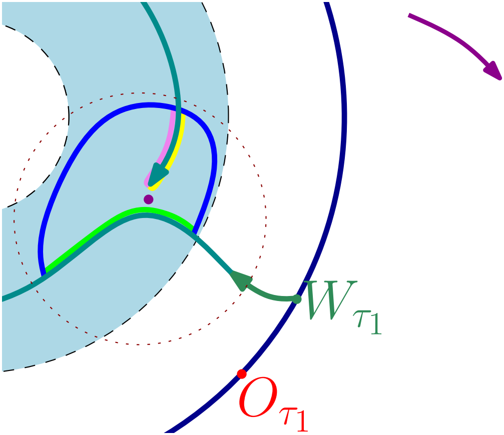

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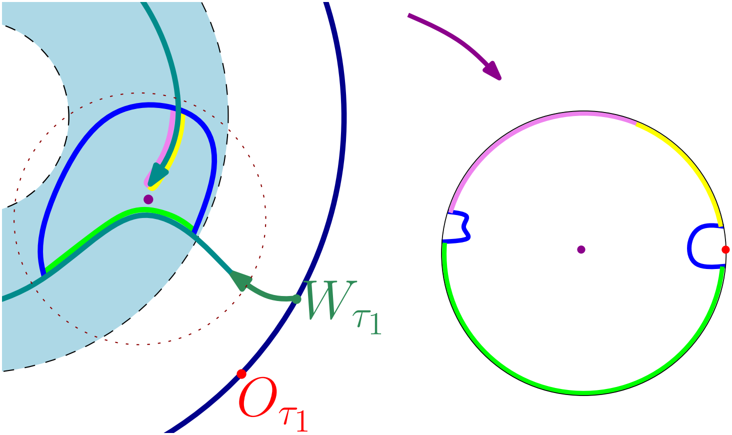

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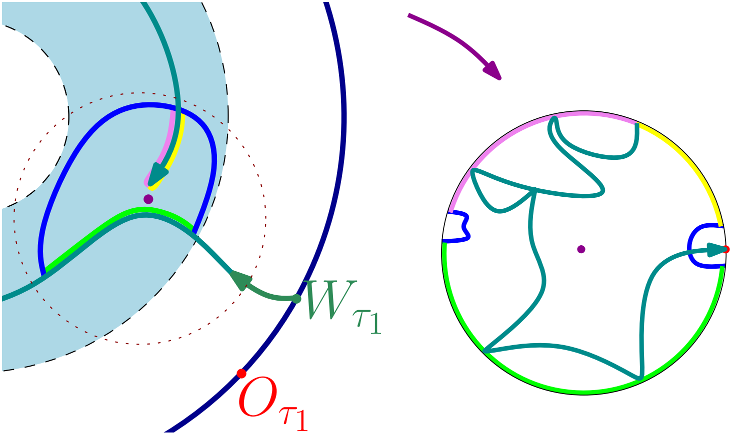

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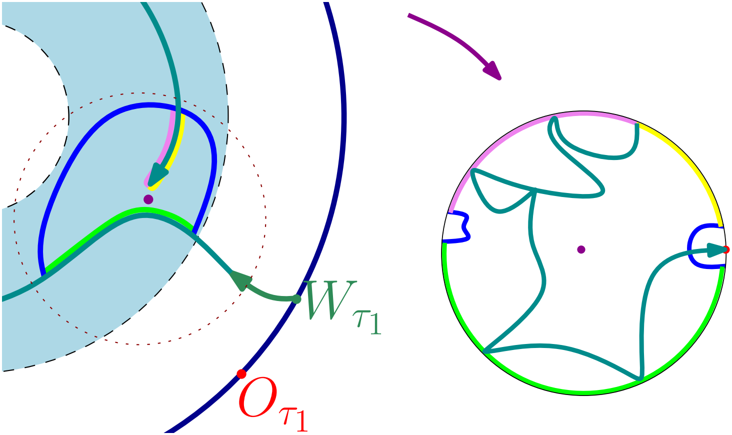

Stage 2 succeeds with positive probability

Conclusion

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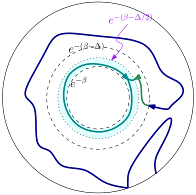
Therefore $\mathbb{P}(E_1^0) = (e^{-\beta})^{\alpha[1+o(1)]}$ ($o(1)$ is in β)



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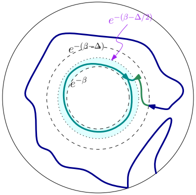

Implies second moment estimate

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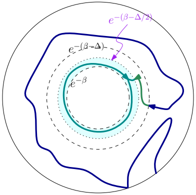

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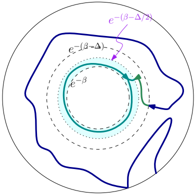

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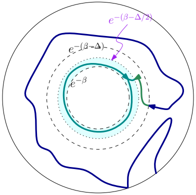

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Taking $\beta \to \infty$ gives $\dim_{\mathcal{H}}(\Gamma) \geq 2 - \alpha$ w.p. 1

Thank you!