Restriction Properties of Annulus SLE

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Introduction

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This means that, given an initial segment and a final segment of a whole-plane SLE_κ curve, the middle part of the whole-plane SLE_κ curve is an annulus $\mathrm{SLE}(\kappa,\Lambda_\kappa)$ curve growing in the complement of the two segments from one tip point to the other tip point.

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The reversibility of annulus SLE(κ, Λ_κ) is related to the reversibility of whole-plane SLE$_\kappa$.

In this talk, I will discuss the restriction properties of the annulus $\mathrm{SLE}(\kappa, \Lambda_\kappa)$ process. Throughout, fix $\kappa \in (0,4]$, let $\mathrm{c} = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$ be the central charge, and let μ_{loop} denote the Brownian loop measure.

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Theorem 1 [Z, 2011]

Let $p>0$, $x,y\in\mathbb{R}$, $a=e^{ix}$, and $b=e^{-p+iy}$. Let β be an annulus SLE(κ,Λ_κ) trace in $\mathbb{A}_p:=\{1>|z|>e^{-p}\}$ from a to b, and let μ denote its distribution. Let L be a relatively closed subset of $\mathbb{A}_p$ such that $\mathbb{A}_p\setminus L$ is a doubly connected domain and contains the neighborhoods of a and b. Define a new probability measure μ_L by

$$\frac{d\mu_L}{d\mu}=\frac{\mathbf{1}_{\beta\cap L=\emptyset}}{Z}\exp(\mathrm{c}\,\mu_{\mathrm{loop}}(\mathcal{L}(\mathbb{A}_p;\beta,L))),$$

where $Z>0$ is some normalization constant, and $\mathcal{L}(\mathbb{A}_p;\beta,L)$ is the set of the loops in $\mathbb{A}_p$ that intersect both β and L. Then μ_L is the distribution of a reparameterized annulus SLE(κ,Λ_κ) curve in $\mathbb{A}_p\setminus L$ from a to b.

If $\kappa = 8/3$, then $\mathrm{c} = 0$. The theorem implies that, β conditioned to avoid L is an annulus $\mathrm{SLE}(\kappa, \Lambda_\kappa)$ trace in $\mathbb{A}_p \setminus L$, up to a reparametrization. For other κ, we get the "weak" restriction property.

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The theorem resembles the restriction theorem for chordal SLE [Lawler-Schramm-Werner, 2003], which says that, if $\mathbb{A}_p$ is replaced by a simply connected domain D, if $D \setminus L$ is also a simply connected domain, and if β is a chordal SLE$_\kappa$ trace, then μ_L is the distribution of a reparameterized chordal SLE$_\kappa$ trace in $D \setminus L$.

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It turns out that the annulus SLE(κ, Λ_κ) process agrees with the annulus SLE constructed by Gregory Lawler recently.

Symbols ($p > 0$):

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Special function ($r > 0$):

$$\mathbf{S}(r, z) = \lim_{M \to \infty} \sum_{k=-M}^{M} \frac{e^{2kr} + z}{e^{2kr} - z}.$$

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Annulus Loewner equation of modulus p driven by $\xi \in C([0,p))$:

$$\partial_t g_t(z) = g_t(z)\mathbf{S}(p - t, g_t(z)/e^{i\xi(t)}), \quad g_0(z) = z.$$

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$$K_t := \{z \in \mathbb{A}_p : \tau_g(z) \le t\}, \quad 0 \le t < p.$$

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Trace (when $\xi(t) = \sqrt{\kappa}B(t) + \text{drift}$):

$$\beta(t) := \lim_{\mathbb{A}_{p-t} \ni z \to e^{i\xi(t)}} g_t^{-1}(z), \quad 0 \le t < p.$$

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4. In the above case, β satisfies $\mathrm{mod}(\mathbb{A}_p \setminus \beta((0, t])) = p - t$. On the other hand, if a simple curve satisfies these properties, then it is an annulus Loewner trace driven by some continuous ξ.

We may lift everything to the covering space. Symbols ($p > 0$):

$$e^i(z) = e^{iz}, \quad \mathbb{S}_p = \{p > \operatorname{Im} z > 0\}, \quad \mathbb{R}_p = \{\operatorname{Im} z = p\}.$$

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Special function ($r > 0$, $\cot_2(z) := \cot(z/2)$):

$$\mathbf{H}(r, z) = -i\mathbf{S}(r, e^i(z)) = \mathrm{P.V.} \sum_{2|n} \cot_2(z - int).$$

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2. $\mathbf{H}(r, \cdot)$ is odd, and has period 2π;
3. $\mathbf{H}(r, z) \in \mathbb{R}$ for $z \in \mathbb{R} \setminus \{\text{poles}\}$, and $\operatorname{Im} \mathbf{H}(r, z) = -1$ for $z \in \mathbb{R}_r$.

Covering annulus Loewner equation of modulus p driven by $\xi \in C([0,p))$:

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Covering trace (when $\xi(t) = \sqrt{\kappa}B(t) + \text{drift}$):

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Relations between (g_t, K_t, β) and $(\widetilde{g}_t, \widetilde{K}_t, \widetilde{\beta})$.

$$g_t \circ e^i = e^i \circ \widetilde{g}_t, \quad \widetilde{K}_t = (e^i)^{-1}(K_t), \quad \beta = e^i \circ \widetilde{\beta}.$$

Another special function ($r > 0$):

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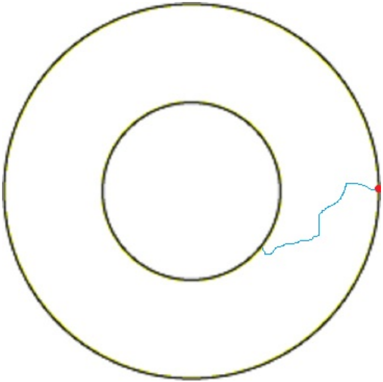
Facts:

1. $\mathbf{H}_I(r,\cdot)$ takes real values on $\mathbb{R}$;
2. If $z \in \mathbb{R}_p$, then $\mathrm{Re}\,\widetilde{g}_t(z)$ satisfies

$$\partial_t \mathrm{Re}\,\widetilde{g}_t(z) = \mathbf{H}_I(p-t, \mathrm{Re}\,\widetilde{g}_t(z) - \xi(t)), \quad 0 \le t < p.$$

Annulus SLE$_\kappa$ (without additional force point) is the annulus Loewner process driven by $\xi(t) = \sqrt{\kappa}B(t)$. The trace starts from 1 on $\mathbb{T}$, and ends at a random point on $\mathbb{T}_p$. The process satisfies DMP.



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1. the force point and the initial point lie on the same boundary component;
2. the two marked points lie on different boundary components.

We now focus on the second case.

Suppose $\Lambda(t,x)$ is C^1 on $(0,\infty)\times\mathbb{R}$, and has period 2π in its second variable. Let $a\in\mathbb{T}$ and $b\in\mathbb{T}_p$. The annulus SLE(κ,Λ) process in $\mathbb{A}_p$ started from a with force point b is defined as follows:

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1. Pick $x,y\in\mathbb{R}$ such that $a=e^{ix}$ and $b=e^{-p+iy}$.
2. Solve the following SDE:

$$d\xi(t)=\sqrt{\kappa}B(t)+\Lambda(p-t,\xi(t)-\operatorname{Re}\widetilde{g}_t^{\xi}(y+ip)),\quad \xi(0)=x$$

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1. Pick $x,y\in\mathbb{R}$ such that $a=e^{ix}$ and $b=e^{-p+iy}$.
2. Solve the following SDE:

$$d\xi(t)=\sqrt{\kappa}B(t)+\Lambda(p-t,\xi(t)-\operatorname{Re}\widetilde{g}_t^{\xi}(y+ip)),\quad \xi(0)=x$$

3. The annulus Loewner process driven by ξ is the annulus SLE(κ,Λ) process to be defined.

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1. The definition does not depend on the choices of x and y because of the periodicity of Λ.
2. For any Λ, the annulus SLE(κ, Λ) process satisfies DMP.
3. In general, the trace may not end at the force point. Even it does, the reversibility may not hold.

It was proved earlier that, if Λ satisfies the PDE:

$$\partial_t \Lambda = \frac{\kappa}{2}\Lambda'' + \left(3 - \frac{\kappa}{2}\right)\mathbf{H}_I'' + \Lambda \mathbf{H}_I' + \Lambda' \mathbf{H}_I + \Lambda' \Lambda, \qquad (1)$$

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then an annulus $\mathrm{SLE}(\kappa; \Lambda)$ process commutes with an annulus $\mathrm{SLE}(\kappa; \Lambda^-)$ process growing in the same domain with the initial point and force point exchanged, where $\Lambda^-(t,x) = -\Lambda(t,-x)$.

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If, in addition, an annulus $\mathrm{SLE}(\kappa; \Lambda)$ trace a.s. ends at the force point, then the reversal of an annulus $\mathrm{SLE}(\kappa; \Lambda)$ trace is an annulus $\mathrm{SLE}(\kappa; \Lambda^-)$ trace, up to some reparametrization. So the reversibility holds.

If we condition an annulus SLE without force point to end at a marked point on $\mathbb{T}_p$, then we get an annulus SLE(κ, Λ) process. The Λ satisfies a different PDE:

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This agrees with (1) only when $\kappa = 2$. For other κ, we need some different method to find a solution of (1).

For $\kappa \in (0,4]$, there is a special drift function Λ_κ which solves (1). Moreover, the annulus SLE$(\kappa;\Lambda_\kappa)$ process satisfies reversibility, and serves as the intermediate process of a whole-plane SLE$_\kappa$ process. Such Λ_κ is defined by the following.

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First, we may transform (1) into a linear PDE using $\Lambda = \kappa\frac{\Gamma'}{\Gamma}$:

$$\partial_t \Gamma = \frac{\kappa}{2}\Gamma'' + \mathbf{H}_I \Gamma' + \frac{6-\kappa}{2\kappa}\mathbf{H}_I' \Gamma. \tag{2}$$

Define a rescaled Jacobi's theta function $\Theta_I(t,z) = \theta_2(\frac{z}{2\pi}, \frac{it}{\pi})$

$$= \prod_{m=1}^{\infty}(1-e^{-2mt})(1-e^{-(2m-1)t}e^{iz})(1-e^{-(2m-1)t}e^{-iz}).$$

Such Θ_I solves $\partial_t \Theta_I = \Theta_I''$, and $\mathbf{H}_I$ can be expressed by $\mathbf{H}_I = 2\frac{\Theta_I'}{\Theta_I}$.

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Let $\Psi=\Gamma\Theta_I^{2/\kappa}$. It is straightforward to check that Γ solves (2) iff Ψ solves another linear PDE ($\sigma=\frac{4}{\kappa}-1$):

$$\partial_t\Psi=\frac{\kappa}{2}\Psi''+\sigma\mathbf{H}_I'\Psi. \qquad (3)$$

We now rescale Ψ. The followings are equivalent:

$$\widehat{\Psi}(t,x) = e^{\frac{x^2}{2\kappa t}} \left(\frac{\pi}{t}\right)^{\sigma+\frac{1}{2}} \Psi\left(\frac{\pi^2}{t}, \frac{\pi}{t}x\right);$$

$$\Psi(t,x) = e^{-\frac{x^2}{2\kappa t}} \left(\frac{\pi}{t}\right)^{\sigma+\frac{1}{2}} \widehat{\Psi}\left(\frac{\pi^2}{t}, \frac{\pi}{t}x\right).$$

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Define another special function $\widehat{\mathbf{H}}_I$ by ($\tanh_2(z):=\tanh(z/2)$)

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One may check that Ψ solves (3) iff $\widehat{\Psi}$ solves another linear PDE:

$$-\partial_t \widehat{\Psi} = \frac{\kappa}{2} \widehat{\Psi}'' + \sigma \widehat{\mathbf{H}}_I' \widehat{\Psi}. \tag{4}$$

As $t \to \infty$, $\widehat{\mathbf{H}}_I \to \tanh_2$, so equation (4) tends to

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which has a simple solution ($\tau = \frac{\kappa}{2} - 2$, $\cosh_2(x) := \cosh(x/2)$):

$$\widehat{\Psi}_\infty(t, x) = e^{-\frac{\tau^2 t}{2\kappa}} \cosh_2^{\frac{2}{\kappa}\tau}(x).$$

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Let $\widehat{\Psi}_q = \widehat{\Psi}/\widehat{\Psi}_\infty$ and $\widehat{\mathbf{H}}_{I,q} = \widehat{\mathbf{H}}_I - \tanh_2$. Then $\widehat{\Psi}$ solves (4) iff $\widehat{\Psi}_q$ solves another linear PDE:

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$$-\partial_t \widehat{\Psi}_q = \frac{\kappa}{2}\widehat{\Psi}_q'' + \tau \tanh_2 \widehat{\Psi}_q' + \sigma \widehat{\mathbf{H}}_{I,q}' \widehat{\Psi}_q. \tag{5}$$

PDE (5) can be solved by a Feynman-Kac formula. Let $X_x(t)$ be a diffusion process which satisfies SDE:

$$dX_x(t) = \sqrt{\kappa} dB(t) + \tau \tanh_2(X_x(t))dt, \quad X_x(0) = x.$$

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One solution of (5) is given by

$$\widehat{\Psi}_q(t,x) = \mathbf{E}\left[\exp\left(\sigma \int_0^\infty \widehat{\mathbf{H}}'_{I,q}(t+s, X_x(s)) ds\right)\right].$$

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It takes some work (using estimation of diffusion processes and Fubini's theorem) to show that $\widehat{\Psi}_q$ is $C^{1,2}$ differentiable. Once this is done, we may apply Itô's formula to show that $\widehat{\Psi}_q$ solves (5).

Let $\widehat{\Psi}_0 = \widehat{\Psi}_\infty \widehat{\Psi}_q$. Then $\widehat{\Psi}_0$ solves (4). Define Ψ_0 using the rescaling rule. Then Ψ_0 solves (3). Let $\Gamma_0 = \Psi_0 \Theta_I^{-2/\kappa}$. Then Γ_0 solves (2). All of these functions are positive. Let $\Lambda_0 = \kappa \frac{\Gamma_0'}{\Gamma_0}$. Then Λ_0 solves (1).

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However, Λ_0 does not have period 2π in its second variable. To fix this problem, we do the following. Let $\Gamma_m(t,x) = \Gamma_0(t, x - 2m\pi)$, $m \in \mathbb{Z}$. Since $\mathbf{H}$ has period 2π in its second variable, every Γ_m also solves the linear PDE (2). Let

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$$\Gamma = \sum_{m \in \mathbb{Z}} \Gamma_m.$$

Some estimations show that the series of functions together with all of their derivatives converge locally uniformly. Thus, Γ also solves (2). The special drift function Λ_κ is defined to be $\Lambda_\kappa = \kappa \frac{\Gamma'}{\Gamma}$.

The annulus SLE(κ, Λ_κ) trace starts from the initial point $a = e^{ix}$, and ends at the force point $b = e^{-p+iy}$. The covering trace starts from x, and may end at $y + 2m\pi + pi$ for some $m \in \mathbb{Z}$. We may decompose this process according to the endpoint of the covering trace.

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Recall that the driving function ξ solves the SDE:

$$d\xi(t) = \sqrt{\kappa} dB(t) + \Lambda_\kappa(p - t, \xi(t) - \mathrm{Re}\, \widetilde{g}_t^{\xi}(y + pi))dt, \quad \xi(0) = x.$$

The drift function Λ_κ is given by $\Lambda_\kappa = \kappa \frac{\Gamma'}{\Gamma}$, where $\Gamma = \sum_{m \in \mathbb{Z}} \Gamma_m$, and $\Gamma_m(t, x) = \Gamma_0(t, x - 2m\pi)$.

Let $y_m = y + 2m\pi$, $m \in \mathbb{Z}$. Suppose ξ_m solves the following SDE:

$$d\xi_m(t) = \sqrt{\kappa}dB(t) + \Lambda_0(p - t, \xi(t) - \mathrm{Re}\, \widetilde{g}_t^{\xi_m}(y_m + pi))dt, \quad \xi(0) = x.$$

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The covering trace driven by ξ_m starts from x and ends at $y_m + pi$, and μ_ξ is a convex combination of the μ_{ξ_m}'s:

$$\mu_\xi = \sum_{m\in\mathbb{Z}} \frac{\Gamma_m(p, x - y)}{\Gamma(p, x - y)} \mu_{\xi_m}.$$

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We call the annulus Loewner process driven by ξ_m a conditional annulus SLE(κ, Λ_κ) process (with initial point x and force point $y_m + pi$).

Theorem 1

Let $p > 0$, $a = e^{ix} \in \mathbb{T}$, and $b = e^{-p+iy} \in \mathbb{T}_p$. Let β be an annulus SLE(κ, Λ_κ) trace in $\mathbb{A}_p$ from a to b, and let μ denote its distribution. Let L be a relatively closed subset of $\mathbb{A}_p$ such that $\mathbb{A}_p \setminus L$ is a doubly connected domain and contains the neighborhoods of a and b. Define a new probability measure μ_L by

$$\frac{d\mu_L}{d\mu} = \frac{\mathbf{1}_{\beta \cap L = \emptyset}}{Z} \exp(\mathrm{c}\, \mu_{\text{loop}}(\mathcal{L}(\mathbb{A}_p; \beta, L))),$$

where $Z > 0$ is some normalization constant, and $\mathcal{L}(\mathbb{A}_p; \beta, L)$ is the set of the loops in $\mathbb{A}_p$ that intersect both β and L. Then μ_L is the distribution of a reparameterized annulus SLE(κ, Λ_κ) curve in $\mathbb{A}_p \setminus L$ from a to b.

Let $p_L = \operatorname{mod}(\mathbb{A}_p \setminus L)$ and $\widetilde{L} = (e^i)^{-1}(L)$. We may find W_L and $\widetilde{W}_L$ such that

$$W_L : (\mathbb{A}_p \setminus L; \mathbb{T}_p) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{A}_{p_L}; \mathbb{T}_{p_L});$$

$$\widetilde{W}_L : (\mathbb{S}_p \setminus \widetilde{L}; \mathbb{R}_p) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{S}_{p_L}; \mathbb{R}_{p_L});$$

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$$W_L \circ e^i = e^i \circ \widetilde{W}_L.$$

Let $\xi(t) = \sqrt{\kappa}B(t)$ plus a drift, $0 \le t < p$, such that $\xi(0) = x$. Let β be the annulus Loewner trace of modulus p driven by ξ, and let $\widetilde{\beta}$ and $\widetilde{g}_t$ be the covering trace and maps. Recall that $\widetilde{\beta}(0) = \xi(0) = x$ and $\beta(0) = e^{ix} = a$.

Let T_L be the first time that $\beta(t) \in L$. If such time does not exist, set $T_L = p$. For $0 \le t < T_L$, let $\beta_L(t) = W_L(\beta(t))$, and

$$v(t) = p_L - \text{mod}(\mathbb{A}_{p_L} \setminus \beta_L((0,t])).$$

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Then β_L is an annulus Loewner trace via the time-change $v(t)$. This means that there exist $\xi_L \in C([0, T_L))$ and two families of conformal maps g_t^L and $\widetilde{g}_t^L$, $0 \le t < T_L$, such that

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$$g_t^L : \mathbb{A}_{p_L} \setminus \beta_L((0,t]) \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{A}_{p_L - v(t)};$$

$$g_t^L \circ e^i = e^i \circ \widetilde{g}_t^L;$$

$$\partial_t \widetilde{g}_t^L(z) = v'(t)\mathbf{H}(p_L - v(t), \widetilde{g}_t^L(z) - \xi_L(t)).$$

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Write $A_j(t) = \widetilde{W}_t^{(j)}(\xi(t))$. From Itô's formula, we have

$$d\xi_L(t) = A_1(t) d\xi(t) + \left(\frac{\kappa}{2} - 3\right) A_2(t) dt.$$

Fix $m \in \mathbb{Z}$. Let $y_m = y + 2m\pi$. Let

$$X_m(t) = \xi(t) - \mathrm{Re}\, \widetilde{g}_t(y_m + pi), \quad X_{L,m}(t) = \xi_L(t) - \mathrm{Re}\, \widetilde{g}_t^L(\widetilde{W}_L(y_m + pi)).$$

$$Y_m(t) = \Gamma_0(p - t, X_m(t)), \quad Y_{L,m}(t) = \Gamma_0(p_L - v(t), X_{L,m}(t)).$$

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Recall that $A_j(t) = \widetilde{W}_t^{(j)}(\xi(t))$. Let

$$A_I(t) = \widetilde{W}_t'(\widetilde{g}_t(y + pi)), \quad A_S(t) = \frac{A_3(t)}{A_1(t)} - \frac{3}{2}\Big(\frac{A_2(t)}{A_1(t)}\Big)^2.$$

So $A_S(t)$ is the Schwarzian derivative of $\widetilde{W}_t$ at $\xi(t)$.

Let $\alpha = \frac{6-\kappa}{2\kappa}$. Define

$$M_m(t) = A_1(t)^\alpha A_I(t)^\alpha \frac{Y_{L,m}(t)}{Y_m(t)} \exp\left(-\frac{c}{6}\int_0^t A_S(s)ds + \alpha \int_{p-t}^{p_L - v(t)} \mathbf{r}(s)ds\right),$$

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where $\mathbf{r}(s)$ is a coefficient in the Laurent expansion of $\mathbf{H}(s,\cdot)$ at 0:

$$\mathbf{H}(s,z) = \frac{2}{z} + \mathbf{r}(s)z + O(z^3).$$

Let $\alpha = \frac{6-\kappa}{2\kappa}$. Define

$$M_m(t) = A_1(t)^\alpha A_I(t)^\alpha \frac{Y_{L,m}(t)}{Y_m(t)} \exp\Big(-\frac{c}{6}\int_0^t A_S(s)ds + \alpha \int_{p-t}^{p_L - v(t)} \mathbf{r}(s)ds\Big),$$

where $\mathbf{r}(s)$ is a coefficient in the Laurent expansion of $\mathbf{H}(s,\cdot)$ at 0:

$$\mathbf{H}(s,z) = \frac{2}{z} + \mathbf{r}(s)z + O(z^3).$$

One may check that $M_m(t)$ is a semi-martingale, and satisfies

$$\frac{dM_m(t)}{M_m(t)} = \Big[\Big(3-\frac{\kappa}{2}\Big)\frac{A_2(t)}{A_1(t)} + A_1(t)\Lambda_0(p_L - v(t), X_{L,m}(t))$$

$$-\Lambda_0(p-t, X_m(t))\Big] \cdot (d\xi(t) - \Lambda_0(p-t, X_m(t))).$$

Suppose now ξ is the solution of

$$d\xi(t) = \sqrt{\kappa} dB(t) + \Lambda_0(p - t, X_m(t))dt, \quad \xi(0) = x. \tag{6}$$

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Then M_m is a local martingale. Since $X_m(t) = \xi(t) - \mathrm{Re}\, \widetilde{g}_t(y_m + pi)$, we see that ξ generates a conditional annulus $\mathrm{SLE}(\kappa, \Lambda_\kappa)$ process, and a.s. $\lim_{t \to p} \widetilde{\beta}(t) = y_m + pi$.

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Girsanov theorem implies that, if $M_m(t)$ is uniformly bounded on $[0, S]$ for some stopping time $S \le T_L$, and if the original probability measure is weighted by $M_m(S)/M_m(0)$, then $\xi(t)$, $0 \le t \le S$, satisfies

$$d\xi(t) = \sqrt{\kappa} d\widetilde{B}(t) + A_1(t)\Lambda_0(p_L - v(t), X_{L,m}(t))dt + \left(3 - \frac{\kappa}{2}\right)\frac{A_2(t)}{A_1(t)}\, dt, \tag{7}$$

where $\widetilde{B}(t)$ is a Brownian motion under the new measure.

Since $d\xi_L(t) = A_1(t)d\xi(t) + (\frac{\kappa}{2} - 3)A_2(t)dt$, we find

$$d\xi_L(t) = A_1(t)\sqrt{\kappa}d\widetilde{B}(t) + A_1(t)^2\Lambda_0(p_L - v(t), X_{L,m}(t))dt.$$

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Since $X_{L,m}(t) = \xi_L(t) - \widetilde{g}_t^L(\widetilde{W_L}(y_m + pi))$ and $v'(t) = A_1(t)^2$, under the new measure, $\beta_L \circ v^{-1}$ up to time $v(S)$ is a conditional annulus SLE$(\kappa; \Lambda_\kappa)$ trace in $\mathbb{A}_{p_L}$. So under the new measure, β up to S is a reparameterized conditional annulus SLE$(\kappa; \Lambda_\kappa)$ trace in $\mathbb{A}_p \setminus L$.

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Let μ_m and $\mu_{L,m}$ denote the distributions of the solutions to (6) and (7), respectively. Then we have

$$\frac{M_m(S)}{M_m(0)} = \frac{d\mu_{L,m}|_{\mathcal{F}_S}}{d\mu_m|_{\mathcal{F}_S}}.$$

We may decompose $M_m(t)$ into the product $M_m(t) = N_m(t)\exp(c\,U(t))$, where $U(t) = \mu_{\mathrm{loop}}(\mathcal{L}(\mathbb{A}_p; \beta((0,t]), L))$. The integral $\int_0^t A_S(s)ds$ is included in the formula for $U(t)$. Let $\mathcal{E}_m$ denote the event that $\lim_{t\to p}\widetilde{\beta}(t) = y_m + pi$, which happens a.s. if ξ solves (6).

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There are two lemmas for $N_m(t)$.

Lemma 1

On the event $\{T_L = p\} \cap \mathcal{E}_m$, we have

$$\lim_{t\to p} N_m(t) = C_{p,p_L},$$

which is a positive constant depending only on p and p_L.

Let $\mathcal{P}_m$ denote the set of (ρ_1, ρ_2) with the following properties.

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1. For $j = 1, 2$, ρ_j is a polygonal crosscut in $\mathbb{S}_p$ that grows from $\mathbb{R}$ to $\mathbb{R}_p$, whose line segments are parallel to either x-axis or y-axis, and whose vertices other than the end points have rational coordinates.

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2. $\rho_1 + 2j\pi$, $\rho_2 + 2k\pi$, $j, k \in \mathbb{Z}$, and $\widetilde{L}$ are mutually disjoint.
3. $\rho_1 \cup \rho_2$ disconnects x and $y_m + pi$ from $\widetilde{L}$ in $\mathbb{S}_p$.

For each $(\rho_1, \rho_2) \in \mathcal{P}_m$, let T_{ρ_1,ρ_2} denote the first time that $\widetilde{\beta}$ hits $\rho_1 \cup \rho_2$. If such time does not exist, set $T_{\rho_1,\rho_2} = p$. Since $\widetilde{\beta}$ starts from x, and $\rho_1 \cup \rho_2$ separates x from $\widetilde{L}$, $T_{\rho_1,\rho_2} \le T_L$.

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Lemma 2

For each $(\rho_1, \rho_2) \in \mathcal{P}_m$, $\ln(N_m(t))$ is uniformly bounded on $[0, T_{\rho_1,\rho_2})$ by a constant depending only on p, L, ρ_1, ρ_2.

Now we study the properties of $U(t)$. We know that U is nonnegative and increasing in t. For any $(\rho_1, \rho_2) \in \mathcal{P}_m$, we have

$$U(T_{\rho_1,\rho_2}) \le \mu_{\text{loop}}(\mathcal{L}(\mathbb{A}_p; e^i(\rho_1) \cup e^i(\rho_2), L)).$$

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Since $\text{dist}(e^i(\rho_1) \cup e^i(\rho_2), L) > 0$, it is shown [Lawler-Werner] that the righthand side is finite. Thus, $U(t)$ is uniformly bounded on $[0, T_{\rho_1,\rho_2}]$.

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Since $M_m(t) = N_m(t)\exp(c\, U(t))$, $M_m(t)$ is uniformly bounded on $[0, T_{\rho_1,\rho_2}]$. So

$$\frac{M_m(T_{\rho_1,\rho_2})}{M_m(0)} = \frac{d\mu_{L,m}|\mathcal{F}_{T_{\rho_1,\rho_2}}}{d\mu_m|\mathcal{F}_{T_{\rho_1,\rho_2}}}.$$

Especially, on the event $\{T_{\rho_1,\rho_2} = p\}$,

$$\frac{d\mu_{L,m}}{d\mu_m} = \frac{M_m(p)}{M_m(0)} = \frac{C_{p,p_L}}{M_m(0)} \exp(c\,\mu_{\rm loop}(\mathcal{L}(\mathbb{A}_p; \beta, L))). \qquad (8)$$

Recall that $\mathcal{E}_m = \{\lim_{t\to p} \widetilde{\beta}(t) = y_m + pi\}$. It is easy to check that

$$\{T_L = p\} \cap \mathcal{E}_m \subset \bigcup_{(\rho_1,\rho_2)\in\mathcal{P}_m} \{T_{\rho_1,\rho_2} = p\}.$$

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On the other hand, since $\mu_{L,m}$ is supported by $\{\beta \cap L = \emptyset\}$, we have

$$\frac{d\mu_{L,m}}{d\mu_m} = \frac{\mathbf{1}_{\beta\cap L=\emptyset}}{M_m(0)} C_{p,p_L} \mu_{\text{loop}}(\mathcal{L}(\mathbb{A}_p; \beta, L)).$$

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Setting $x_L = \widetilde{W}_L(x)$ and $y_L = \mathrm{Re}\, \widetilde{W}_L(y + pi)$, we get

$$M_m(0) = \frac{\Gamma_m(p_L, x_L - y_L)}{\Gamma_m(p, x - y)}.$$

Recall that if μ is the distribution of the driving function of an annulus SLE(κ, Λ_κ) trace in $\mathbb{A}_p$ from e^{ix} to e^{-p+iy}, then

$$\mu = \sum_{m \in \mathbb{Z}} \frac{\Gamma_m(p, x-y)}{\Gamma(p, x-y)}\, \mu_m.$$

Recall that if μ is the distribution of the driving function of an annulus $\mathrm{SLE}(\kappa, \Lambda_\kappa)$ trace in $\mathbb{A}_p$ from e^{ix} to e^{-p+iy}, then

$$\mu = \sum_{m \in \mathbb{Z}} \frac{\Gamma_m(p, x-y)}{\Gamma(p, x-y)} \, \mu_m.$$

Similarly, if μ_L is the distribution of the driving function of an annulus $\mathrm{SLE}(\kappa, \Lambda_\kappa)$ trace in $\mathbb{A}_p \setminus L$ from e^{ix} to e^{-p+iy}, then

$$\mu_L = \sum_{m \in \mathbb{Z}} \frac{\Gamma_m(p_L, x_L - y_L)}{\Gamma(p_L, x_L - y_L)} \, \mu_{L,m}.$$

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$$\mu_L = \sum_{m\in\mathbb{Z}} \frac{\Gamma_m(p_L,x_L-y_L)}{\Gamma(p_L,x_L-y_L)}\,\mu_{L,m}.$$

Therefore,

$$\frac{d\mu_L}{d\mu} = \frac{\Gamma(p_L,x_L-y_L)}{\Gamma(p,x-y)} C_{p,p_L}\mathbf{1}_{\beta\cap L=\emptyset}\exp(c\,\mu_{\mathrm{loop}}(\mathcal{L}(\mathbb{A}_p;\beta,L))),$$

which finishes the proof with $Z=\Gamma(p,x-y)/(\Gamma(p_L,x_L-y_L)C_{p,L})$.

The following theorem connects annulus SLE(κ, Λ_κ) with chordal SLE$_\kappa$. This theorem shows that the annulus SLE(κ, Λ_κ) process agrees with the annulus SLE$_\kappa$ defined by Lawler.

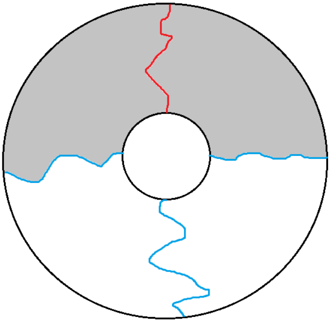
Theorem 2

With other conditions the same as in Theorem 1 except that now $\mathbb{A}_p \setminus L$ is a simply connected domain, μ_L is the distribution of a reparameterized chordal SLE$_\kappa$ trace in $\mathbb{A}_p \setminus L$ from a to b.

Theorem 2 can be used to construct multiple disjoint SLE_κ curves crossing an annulus.

Definition

Let $n \geq 2$. Let $a_1, \ldots, a_n$ (resp. $b_1, \ldots, b_n$) be distinct points on $\mathbb{T}$ (resp. $\mathbb{T}_p$) that are oriented counterclockwise. A random n-tuple of disjoint curves $(\beta_1, \ldots, \beta_n)$ is called a multiple SLE_κ in $\mathbb{A}_p$ from $(a_1, \ldots, a_n)$ to $(b_1, \ldots, b_n)$, if for any $j \in \{1, \ldots, n\}$, conditioned on all other $n-1$ curves, β_j is a chordal $\mathrm{SLE}(\kappa)$ trace from a_j to b_j that grows in D_j, which is the subregion in $\mathbb{A}_p$ bounded by β_{j-1} and β_{j+1} ($\beta_0 := \beta_n$ and $\beta_{n+1} := \beta_1$) that has a_j and b_j as its boundary points.



The picture shows multiple SLE with $n = 4$. When the 3 blue curves are known, the red curve is a chordal SLE_κ that grows in the grey region.

The following result resembles the work by Kozdron and Lawler for simply connected domains.

Corollary

Let $\mathbb{A}_p, n, a_j, b_j$ be as in the definition. For $1 \le j \le n$, let β_j be an annulus $\mathrm{SLE}(\kappa, \Lambda_\kappa)$ curve in $\mathbb{A}_p$ from a_j to b_j, and let μ_{β_j} denote its distribution. Define a new probability measure μ^M by

$$\frac{d\mu^M}{\prod_j \mu_{\beta_j}} = \frac{\mathbf{1}_{\mathcal{E}_{\mathrm{disj}}}}{Z} \exp\Big(\mathrm{c} \sum_{k=1}^{n} (k-1)\mu_{\mathrm{loop}}(\mathcal{L}_k)\Big), \tag{9}$$

where $Z > 0$ is some normalization constant, $\mathcal{E}_{\mathrm{disj}}$ is the event that β_j, $1 \le j \le n$, are mutually disjoint, and $\mathcal{L}_k$ is the set of loops in $\mathbb{A}_p$ that intersect at exactly k curves among $\beta_1, \dots, \beta_n$. Then μ^M is the distribution of a multiple SLE_κ in $\mathbb{A}_p$ from $(a_1, \dots, a_n)$ to $(b_1, \dots, b_n)$.

Proof. Fix $j \in \{1, \ldots, n\}$. Let $\mathcal{E}^j_{\text{disj}}$ denote the event that β_k, $k \neq j$, are mutually disjoint. When $\mathcal{E}^j_{\text{disj}}$ occurs, let D_j be as in the definition, and $L_j = \mathbb{A}_p \setminus D_j$. The key step is that the righthand side of (9) can be written as

$$C_j \mathbf{1}_{\{\beta_j \cap L_j = \emptyset\}} \exp(\mathrm{c}\, \mu_{\text{loop}}(\mathcal{L}(\mathbb{A}_p; \beta_j, L_j))),$$

where C_j is measurable w.r.t. the σ-algebra generated by β_k, $k \neq j$.

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where C_j is measurable w.r.t. the σ-algebra generated by β_k, $k \neq j$.

Let μ_j^M denote the conditional distribution of β_j when $(\beta_1, \ldots, \beta_n) \sim \mu^M$ and all β_k other than β_j are given. Then

$$\frac{d\mu_j^M}{d\mu_j}\Big|_{\{\beta_k : k \neq j\}} = C_j \mathbf{1}_{\{\beta_j \cap L_j = \emptyset\}} \exp(\mathrm{c}\, \mu_{\text{loop}}(\mathcal{L}(\mathbb{A}_p; \beta_j, L_j))).$$

From Theorem 2 we conclude that μ_j^M is the distribution of a time-change of a chordal SLE(κ) trace in $\mathbb{A}_p \setminus L_j = D_j$ from a_j to b_j. $\square$

Thank you!