Dimers and families of Cauchy-Riemann operators

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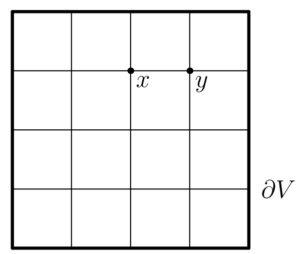
Notes taken by Samuel S Watson

1 Discrete free field

Consider a graph $\Gamma = (V, E)$, with some portion of the graph designed as boundary.



We consider (discrete) one-forms $\omega : E \to \mathbb{R}$ for which $\omega(\overrightarrow{xy}) = -\omega(\overrightarrow{yx})$. We can put a norm on the linear space of such one-forms, namely

$$\|\omega\|^2 = \sum_{x\sim y} c_{xy}|\omega(\overrightarrow{xy})|^2, \qquad c_{xy} > 0.$$

The DGFF is obtained from a random one-form with density

$$e^{-\frac{1}{2}\|\omega\|^2}\, d\lambda_{\text{closed}},$$

where $d\lambda_{\text{closed}}$ is Lebesgue measure on the set of closed one-forms, i.e., those for which $\sum \omega(\overrightarrow{x_i x_{i+1}})$ for the vertices $\{x_i\}$ around a face is zero for every face. The Gaussian free field is defined to be

$$h(x) = \int_{\text{boundary}}^{x} \omega,$$

where the integral refers to a summation of the value of a one-form along a path from a boundary point to x. Note that the integral does not depend on which point on the boundary is chosen, because of the requirement that ω is closed. Equivalently, h has Gaussian density

$$\exp\left(-\frac{1}{2}\sum_{x,y} c_{xy}(h(x) - h(y))^2\right)\prod dh(x).$$

The simplest observables we can consider are the electric correlators. For example, taking vertices x and y in the graph,

$$\mathbb{E}e^{i\alpha(h(x)-h(y))} = \exp\left(-\frac{\alpha^2}{2}(\operatorname{Var} h(x) + \operatorname{Var} h(y) - 2\operatorname{Cov}(h(x), h(y)))\right).$$

More generally,

$$\mathbb{E}e^{i\sum \alpha_j h(x_j)} = \exp\left(-\frac{1}{2}\sum_{j,k} \alpha_j\alpha_k \operatorname{Cov}(h(x_j)h(x_k))\right),$$

where the charges α_j sum to zero so we don't have to worry about boundary conditions. By a discrete version of Green's formula, we can write the Dirichlet energy of h as

$$\sum c_{xy}(h(x) - h(y))^2 = \sum_x h(x)(\Delta h)(x),$$

where Δh is the usual discrete Laplacian $\Delta h(x) = \sum_{y\sim x} c_{xy}(h(x) - h(y))$ (positive convention for the Laplacian). Therefore, we see that the covariance is the inverse of the operator Δ^{-1}.

$$\operatorname{Cov}(h(x), h(y)) = \Delta^{-1}(x, y).$$

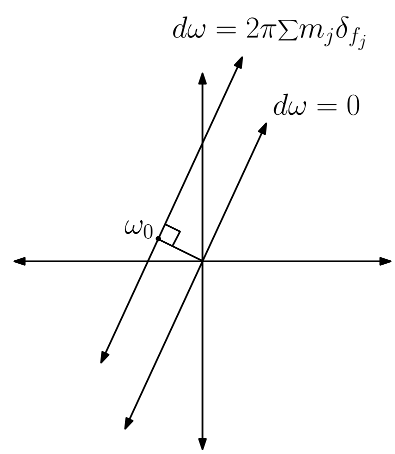
Now consider placing some magnetic charges m_j at some of the faces, so that instead of the space of closed one-forms we are considering an affine space

$$\{\omega : d\omega \equiv 2\pi \sum m_j \delta_{f_j}\},$$

where f_j is the face at which the magnetic charge m_j is located. We again consider the energy $e^{-\frac{1}{2}\|\omega\|^2} d\lambda$, where this time λ is Lebesgue measure on the affine space. In this case, h becomes multivalued, because integration of ω around a magnetic pole causes us to pick up extra constants. Then the partition function for this model (divided by the usual partition function) gives

$$\frac{\int e^{-\frac{1}{2}\|\omega\|^2} d\lambda_{d\omega=2\pi\sum m_j\delta_{f_j}}}{\int e^{-\frac{1}{2}\|\omega\|^2} d\lambda_{d\omega=0}} = \exp\left(-\frac{1}{2}\|\omega_0\|^2\right),$$

where ω_0 has minimal energy $\|\cdot\|^2$ among the set of one forms above. Note that we may think of $\{\omega : d\omega \equiv 2\pi \sum m_j \delta_{f_j}\}$ as an affine space (see figure below), in which case ω_0 plays the role of the orthogonal projection of the origin to this space.



Since the Dirichlet energy is minimized by harmonic functions, ω_0 is characterized among elements in the affine space by a certain natural "harmonicity" requirement:

$$d\omega_0 = 2\pi \sum m_j f_j$$
$$d * \omega_0 = 0,$$

where the second equation is dual to harmonicity for functions (the operator is a discrete version of the Hodge star operator).

For the continuum Gaussian free field, we define h as a distribution with covariance operator

$$\mathrm{Cov}(h(x), h(y)) = G(x, y),$$

where G is the Green's kernel of Δ with Dirichlet boundary conditions. Note that this is a natural limit of the discrete model, because the simple random walk giving the discrete Green kernel will converge to Brownian motion, giving the continuous Green kernel. One may regard h as a random element of $H^{-\epsilon}(D)$ for any $\epsilon > 0$. The characteristic function would, if the field were continuous, be given by

$$\mathbb{E}\exp\left(i\sum \alpha_j h(x_j)\right) = \exp\left(-\frac{1}{2}\sum_{j,k} \alpha_j\alpha_k G(x_j, x_k)\right).$$

However, G is infinite when $j = k$. To deal with this singularity, we replace h with its average on a disk of radius δ (a "smeared" version of h). We get

$$\mathbb{E}\exp\left(i\sum \alpha_j \frac{1}{\pi\delta^2}\int_{D(x_j,\delta)} h(x)\, d\lambda\right) = \delta^{\text{some power}} \exp\left(-\frac{1}{2}\sum_{j,k} \alpha_j\alpha_k \tilde{G}(x_j, x_k)\right),$$

where $\tilde{G}(x, y) = \lim_{y\to x} G(x, y) - \frac{1}{2\pi}\log|x-y|$ when $x = y$ and is equal to the usual Green's function otherwise. Thus if we normalize by the singularity which has now been isolated in the power of δ, we get a well-defined correlator.

What happens when we put magnetic charge into the picture? We again put some singularities $f_1, f_2, \ldots, f_n$ in the domain and associate magnetic charges to each of them. We have $h = h_{\mathrm{Dir}} + h_0$, where h_{Dir} is the GFF with Dirichlet boundary conditions and h_0 is a multivalued function of minimal energy in the space of functions with magnetic charges of the given values and locations. Again, the ratio of the partition function of this model to the partition function of the charge-free model should be the energy of of h_0, namely $\exp\left(-1/2\|h_0\|_\nabla^2\right)$. Explicitly, the requirements on h_0 are that

$$\Delta h_0 = 0$$

away from insertions and

$$h_0(x) = m_j \operatorname{Im}\log(x - f_j),$$

near an insertion f_j. So the Dirichlet energy of h_0 should be $\int_{\text{reg}} |\nabla h_0|^2$, but near a singularity $|\nabla h_0|$ is proportional to the inverse distance to f_j, which is not integrable. We can regularize by subtracting off the logarithmic divergence:

$$\int_{\text{reg}} |\nabla h_0|^2 = \int_{\text{complement of } \delta \text{ disks around singularities}} |\nabla h_0|^2 - c\log\delta,$$

which gives us something well-defined.

In the whole plane, we specify the action

$$S(\phi) = \frac{g}{4\pi}\int |\nabla\phi|^2,$$

where g is the *coupling constant*. We may define the regularized characteristic function (requiring $\sum \alpha_j = 0$ because we're working in the whole plane), we get

$$\mathbb{E}^{\text{reg}} \exp\left(i \sum \alpha_j h(x_j)\right) = \prod_{j,k} |x_j - x_k|^{g^{-1}\alpha_j,\alpha_k},$$

where we require $\sum \alpha_j = 0$. Similarly, for magnetic charge m_j at x_j we have

$$\langle O_{m_j}(x_j)\rangle = \prod_{j,k} |x_j - x_k|^{g m_j m_k},$$

where again $\sum m_j = 0$.

Instead of looking at a scalar field, we may look at a compactified field

$$h : D \to \mathbb{R}/2\pi r\mathbb{Z},$$

where r is the *radius of compactification*. Suppose for example that D is a torus (because for simply connected domains it the compactification isn't so useful). We may decompose

$$h = \overbrace{h_s}^{\text{scalar free field}} + \overbrace{h_{\text{insertions}}}^{\text{harmonic, multivalued}},$$

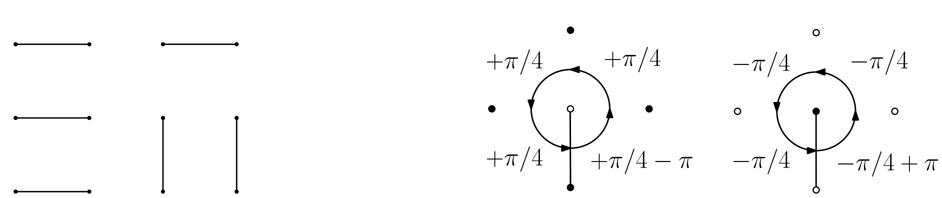
as before. Recall that if ω is a closed one-form on the torus, then ω can be composed uniquely in the form $dh + (a\,dx + b\,dy)$, a sum of an exact one-form an a harmonic one-form. This decomposition is orthonormal with respect to the Dirichlet energy, i.e.,

$$\|\omega\|^2 = \|dh\|^2 + \|a\,dx + b\,dy\|^2$$

We request that $\int a\,dx + b\,dy \in 2\pi r\mathbb{Z}$ whenever the domain of integration is a nontrivial closed loop.

2 Dimers

Recall that a dimer cover is a collection of edges in a graph G (which we take to be a subgraph of $\mathbb{Z}^2$) which covers each vertex exactly once. Recall that $\mathbb{Z}^2$ is bipartite: we may partition the vertices into two sets with all edges going from one set to the other (in checkerboard fashion). Associated with a dimer cover we associate a height function on faces as illustrated in the figure below. In words, given the value of a function on a face, we obtain the value of the function on an adjacent face by moving counterclockwise about a black vertex and increasing by $\pi/4$ for an unmatched edge or increasing by $\pi/4 - \pi$ for a matched edge (signs are reversed if we pivot about a white vertex).



As the mesh size goes to zero, this height function converges to the Gaussian free field. The idea of this proof is to study the correlations between height differences across edges. The coupling constant obtained is $g = 2$.

How do we incorporate electric and magnetic charges in this picture? It isn't clear from the weakest forms of convergence that we can still extract information about singular correlators. For example, we might hope that

$$\langle e^{i\sum \alpha_j h(x_j)}\rangle = c\prod_{j<k} |x_j - x_k|^{g^{-1}\alpha_j\alpha_k}.$$

If we had convergence strong enough to replace h with an average of the GFF on a face of $\epsilon\mathbb{Z}^2$, we would obtain this formula from our discrete calculations. This is too much to ask, however, because of the periodicity of the left-hand side (which follows from the discrete-valued nature of the dime height function h). The best we can do is $\alpha_j \in (-1, 1)$. This is maximal because the LHS is periodic with period 2 in each charge.

Proposition 1. Correlators for the dimer height function converge to their GFF counterparts up to the point at which the formula has to break down.

At the moment this is only for the plane, but you could do it with Temperleyan boundary conditions if you wanted.

What is the analogue of a magnetic charge in a dimer model? We consider defects in our graph (interior vertices which we remove), and we consider only dimer covers of the non-defect vertices. Because of the way the dimer height function is defined, the lack of a matching edge at a white defect results in an increase of $+\pi$ for h at that vertex. Similarly, at a white vertex h is decreased by π. So we see that the magnetic charges are $m = \pm 1/2$, and we should have

$$\langle O_{1/2}(b_1)\cdots O_{1/2}(b_n)O_{-1/2}(w_1)\cdots O_{-1/2}(w_n)\rangle \sim |b_i - b_j|^{1/2}|w_i - w_j|^{1/2}|b_i - w_j|^{-1/2}.$$

In fact, it is possible to make this statement rigorous and prove it.

The dimer model on the torus corresponds to the compactified free field, and we obtain coupling constant $g = 2$ and radius of compactification $r = 1/2$.

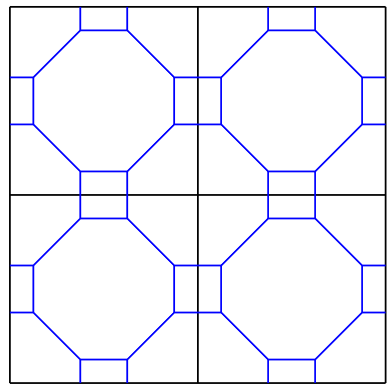
3 Ising bosonization

Ising bosonization refers to a collection of identities which relate Ising correlations with electric and magnetic bosonic correlations (Itzykson-Zuber; D Francesco, Saleur, Zuber). Consider the spin Ising model on a subgraph of $\mathbb{Z}^2$ (we associate ± 1 spins with each face, and these interact in a ferromagnetic way), and consider the spin correlation relationship

$$\langle \sigma(x_1)\cdots\sigma(x_n)\rangle^2 \asymp \left\langle \prod_j \left(e^{i\phi(x_j)} + e^{-i\phi(x_j)}\right)\right\rangle,$$

where the right-hand side consists of electric correlators of the free field.

Given some planar graph, define the graph illustrated in blue below, which is bipartite even if the original one is not.



We can prove that all the bosonization identities are exact on a combinatorial level if we replace the free field h with the height function h on this blue graph.