Lax representation for the non-linear sigma model with a global U(1)×U(1) isometry

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Notes taken by Samuel S Watson

We consider a non-linear sigma model with action

$$S = \frac{1}{2\pi}\int d^2x(\eta^{ab}G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \cdots).$$

Recall that the Ricci flow is defined by

$$\partial_\tau G_{\mu\nu} = -R_{\mu\nu},$$

where $\tau = -1/2\pi \times \log$(energy scale). For example, on the sphere, the Ricci flow causes the radius of the sphere to shrink and collapse to a point.

Some approaches include S-matrix bootstrap, Coordinate Bethe ansatz, Algebraic Bethe ansatz, Hirota type difference relations. We can make various assumptions, including lattice assumptions, cutoffs, scaling limits or the string hypothesis and/or certain analytical assumptions.

To play with the cut-off free quantum integrability one needs to have an efficient nonperturbative control over the UV behavior. With a proper uniformization (certain choice of λ it should provide a sufficiently large domain of analyticity for the quantum transfer-matrices.

So our problem is the construct Lax representations for sigma models possessing UV finite, multi-parametric RG/Ricci flow.

Let (M, G, ∇) be a Riemann-Cartan N-manifold equipped with a metric and an affine connection ∇, satisfying

$$\nabla G = 0$$
$$H_{\mu\nu\sigma} = G_{\mu\rho}(\Gamma^\rho_{\nu\sigma} - \Gamma^\rho_{\sigma\nu}) \in \Lambda^3(T^*M_N).$$

In order to build a flat connection, we need to impose additional conditions for poles of $F(\lambda)$ and $F(0) = 0$.

We can use meromorphic functions to uniformize the squashed sphere.

In review, we found Lax representations for a 4-parametric family of classically integrable σ-model which includes $SU(2)$ principal chiral field with WZW term, anisotropic $SU(2)$ principal chiral field with WZW term, Fateev 3D sausage, and Fateev, Onofri, Al Zamolodchiov 2D sausage.

We also described 4-parametric family of solutions of the Ricci flow which generalized the Fateev 2-parametric family.