Taming the Integrable Zoo

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MSRI workshop on conformal invariance and statistical mechanics

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Notes taken by Samuel S Watson

Overview: we're going to talk about the relationship between various stories: integrability and the Yang-Baxter equation, knot and link invariants, discrete holomorphicity and lattice operators. We will use discrete holomorphicity to turn topological invariants into integrable Boltzmann weights.

Remark: integrability plays a key role in making the connection between lattice models and SLE.

1 Yang-Baxter

The Yang-Baxter equation is a functional equation specifying a condition on the Boltzmann weights (essentially the model definition) necessary for the model to be integrable.

We consider a particular $O(n)$ loop model, called the completely packed loop model, or Q-state Potts model. The partition function is

$$Z = \sum d^{n_l} v^{n_v} h^{n_h},$$

where d is the weight per loop, v (h) is the weight per vertical (horizontal) avoidance. The main feature of this model is that all the edges are involved in some loop.

We can rewrite the partition function

$$Z = \sum_{\text{graphs}} (\text{topological weight}) \times (\text{local weights}),$$

in every critical model (and possibly in others). The Yang-Baxter equation says that sums of products of three Boltzmann weights must obey an equation (see figure). In the case of our example model, setting $w(u) = h(u)/v(u)$, the YB equation becomes

$$w(u)w(u+u')w(u') + d\, w(u)w(u') + w(u)w(u') - w(u+u') = 0,$$

which is about as simple as it gets for Yang-Baxter. There is a solution here, which is

$$w(u) = \frac{q^{-1}e^{iu} - qe^{-iu}}{e^{iu} - e^{iu}}.$$

2 Knots

A knot invariant depends only on the topology of the knot, so it can be used to distinguish knots. The Jones polynomial works by projecting the knot into two dimensions, and then we replace every

over/undercrossing with a formal expression involving $q^{1/2}$ and $q^{-1/2}$. Closed loops get a weight $-q - q^{-1}$. Solutions of this can be found by taking a limit of the Yang-Baxter equation. To make a topological invariant, we have to keep track of the number of twists, called the *writhe*. From the point of view of knot theory, this seems like a technical annoyance. For us, however, it will turn out to be the thing that provides a connection to discrete holomorphicity.

3 Discrete holomorphicity

An operator $O(z)$ in some two dimensional lattice model is called discrete holomorphic if its expectation values obey the lattice Cauchy Riemann equations, i.e., around a closed path we have

$$\sum O(z_i)\,\delta z_i = 0.$$

Smirnov et al proved discrete holomorphicity for certain models.

Cardy et al switched the perspective: leave the Boltzmann weights undetermined, posit the existence of some discrete holomorphic operator. This imposes linear constraints on the Boltzmann weights, and these turn out in some cases to be the same constraints as the Yang-Baxter equation.

4 Relationship between discrete holomorphicity and integrability

Note that solutions of the YBE depend on the parameter u. One must "Baxterize" the knot invariant to obtain the Boltzmann weights. We will see that knot invariants + braiding gives integrability, loosely speaking.

The key connection comes from looking at the way the discretely holomorphic operators are defined. They have a "string" attached, for example a path from the point at which the operator operates to the boundary. The expectation value $\langle O(z)\rangle$ is independent of the string's path except for the total winding angle: you have to pick up an extra factor $e^{2\pi i h}$ if you go around one more time. This is also characteristic of the ribbon-twisting mentioned before.

The same mathematical structure appears under many different titles: modular tensor category, topological quantum field theory, rational conformal field theory, consistent braiding and fusing relations for anyons.

An anyon is a particle in two spatial dimensions with statistics generalizing bosons and fermions. Exchanging two particles gives a braid in 3D spacetime. These follow the same rules as for the Jones polynomial, as well as some extra rules specifying behavior under *fusion*, where two particles merge, among other things. In the abelian case, fusion is non-trivial but doable. In the non-abelian case, it's more work but we can do it using quantum groups.

5 Bringing it all together

A discrete holomorphic operator is defined by modifying the topological part.

$$\langle \psi(z)\psi(w)\rangle = \sum_{\text{graphs}} \text{Eval(topological part)} \times \text{(local weights)}.$$

For this operator to be discretely holomorphic, the Boltzmann weights must obey the lattice Cauchy-Riemann equation. This linear relation may be rewritten in terms of three graphs (see slides). Since all the coefficients must vanish, we get three linear equations and one unknown. Nevertheless, there is a solution (and of course it's the same solution we had before).

This observation allows you to see that the $\mathfrak{u}$ in the Boltzmann weights really is the angle.

6 Applications

This approach provides a systematic way of understanding known cases and allows generalization of known discrete holomorphic operators.

Also, we can study height variants of these loop models. Finally, it's useful to incorporate topology in an illuminating way.

7 Next steps

Combine holomorphic operators with anti-holomorphic operator to get a lattice analogue of CFT primary fields.

Is it possible that an integrable model be found for each anyon theory? Each has multiple vertices; can a model be found for each vertex? The answer to the first question is probably yes; the second we don't know one way or the other.

Is the full modular tensor category structure necessary?

May enable more of the "nicer" discrete holomorphic observables, aiding in the attempt to prove conformal invariance (i.e., a substitute for the nice properties of the Ising model that permitted proof in that case.)

Is it possible to generalize SLE? Much recent work suggests that there should be some generalization of SLE (Cardy's work on discrete holomorphic observables, etc).

Why does SLE apply to integrable lattice models? What does the geometry from this approach have to do with integrability?

Finally, what really is the reason that integrable models work?