Zeros of Gaussian analytic functions—invariance and rigidity

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A point process is a random configuration of points in a space such as $\mathbb{R}^d$. Equivalently, a point process is a random discrete measure.

The most well studied example is the Poisson point process, though it's important to emphasize that there are many other models, not related to this one, which are important and have been studied by physicists.

One limitation of PPP is the lack of spatial correlations. We look for new models, such as the Ginibre model (take a large matrix of complex Gaussians $N_{\mathbb{C}}(0,1)$, and look at the eigenvalues and let the size of the matrix go to infinity with no normalization – not a Hermitian matrix, so the eigenvalues aren't real) and the zeros of Gaussian analytic functions. These models have some natural repulsion, meaning that points tend not to clump.

We say that a sequence point processes μ_n converges to μ if $\int \phi d\mu_n \to \int \phi \, d\mu$ for all $\phi \in C_c(\mathbb{C})$.

In the Ginibre ensemble, we get a translation invariant distribution by following the above-described procedure.

For the second model, we define

$$f_n(z) = \xi_0 + \xi_1/\sqrt{1!}z + \cdots + \xi_n/\sqrt{n!}z.$$

Consider the set ν_n of zeros of f_n and let $n \to \infty$. It turns out that this set gives the zeros of the analytic function $f(z) = \sum \xi_k/\sqrt{k!}z^k$. The resulting process is translation invariant and ergodic.

Theorem 1. (Sodin rigidity) $f(z)$ is the unique Gaussian entire function with a translation invariant zero process of intensity 1.

A Gaussian analytic function is one for which $(f(z_1), \ldots f(z_k))$ is a jointly normal vector, for every collection of points $\{z_i\}$.

We say that π is deletion tolerant if with probability 1 we fail to detect the change when we delete all points of π from a bounded domain, i.e., $\tilde{\pi} \ll \pi$.

Example. Let $f(z) = a_n z^n$ with a_n iid Gaussians. This function is only analytic in the disk, and the zeros of this function are deletion tolerant in the sense that removing the zeros in a finite hyperbolic-area region gives a measure which is absolutely continuous with respect to the original process (i.e., you can't tell something is wrong).

We consider the conditional law of the points inside a disk given the points outside the disk.

Theorem 2. (Ghosh, Nazarov, P., Sodin, 12)
(i) The set ω of points outside $\mathbb{D}$ determine exactly the number of points inside $\mathbb{D}$, "and nothing more:" (ii) For almost every ω, the conditional measure $\rho_\omega(\zeta)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{D}^{N(\omega)}$.

Moreover, bounds can be determined for the derivative, in terms of Vandermonde determinants.

We say that a process is *rigid* if with probability 1 we can determine exactly how many zeros are inside the disk, given the points outside the disk.

Theorem 3. (Ghosh, Nazarov, P., Sodin, 12)
(i) For the GAF zero ensemble, the set ω of points outside $\mathbb{D}$ determine exactly the number of points inside $\mathbb{D}$, and their sum, "and nothing more:" (ii) For almost every ω, the conditional measure $\rho_\omega(\zeta)$ is absolutely continuous with respect to Lebesgue measure on $\sigma_{S(\omega)}$, i.e., the set of points with the correct center of mass.

For example, if there are two points, they can be anywhere such that they're both in the domain and their midpoint is in the correct location.

We say that π is rigid at level k if the points outside $\mathbb{D}$ determine 0, 1, ..., k-1 moments of the points in $\mathbb{D}$. Also, we insist that the conditional distribution of the points inside given the configuration outisde has nonvanishing density with respect to Lebesgue measure on the submanifold of $\mathbb{D}^{N(\omega)}$ defined by 0, 1, ..., $k-1$ moments being conserved.

Poisson is Level 0, Ginibre is level 1, GAF zeros is at level 2. Are there natural processes for higher levels?

The rigidity of the number of points of the GAF is the easiest part. The idea is to use a calculation of Sodin and Tsirelson, (letting $\varphi_L(z) = \varphi(z/L)$), we have

$$\mathrm{Var} \int \phi_L \, d\nu = O(L^{-2}).$$

Let φ be roughly $1_{\mathbb{D}}$ and C_c^2. By Sodin Tsirelson,

$$\int \varphi_L \, d\nu \approx \mathbb{E} \int \varphi_L \, d\nu = \int \varphi_L(z) \rho_1(z) \, dm(z).$$

But $\int \varphi_L \, d\nu = n(\mathbb{D}) + \int_{\mathbb{D}_L \setminus \mathbb{D}} \varphi_L \, d\nu$. This means we can compute $n(\mathbb{D})$ approximately; now let $L \to \infty$.

Let's compare properties of this function to $f(z) = \sum a_n z^n$.

Theorem 4. (Hannay, Zelditch-Shiffman, etc) The law of the zeros of f is invariant under Möbius transformations preserving the disk.

Note that, letting $f = \sum a_n z^n / \sqrt{n!}$, we have

$$\mathrm{Cov}(f(z), f(w)) = \sum \frac{z^n \overline{w}^n}{n!} = e^{z\overline{w}},$$

because many terms in $\mathbb{E}\sum_n \frac{a_n z^n}{\sqrt{n!}}\sum_k \frac{a_k z^k}{\sqrt{k!}}$ cancel. From this point of view, we can see why $1/\sqrt{n!}$ is natural from the point of view of trying to characterize the process by covariance structure. Note that

$$\mathrm{Cov}[f(z+a), f(w+a)] = e^{z+a}e^{\overline{w}+\overline{a}},$$

which means that the zeros of the original function has a translation invariant distribution.

We define $p_\epsilon(z_1,\dots,z_n)$ to be the probability that a random function f has zeros in the ϵ balls around every z_i. The joint intensity of zeros is defined by the limit of this probability normalized by $(\pi\epsilon^2)^n$, if this limit exists.

One may calculate this probability for a Gaussian analytic function in a planar domain D. Also, the joint intensity of zeros for the disk GAF is given by the Berman kernel $\det[K(z_i, z_j)]$. This makes available the tools of the well-developed theory of determinantal processes. One consequence of this connection is

Theorem 5. Let X_k be 1 with probability r^{2k} and 0 with probability $1 - r^{2k}$. Then $\sum_{k=1}^{\infty} X_k$ and N_r = the number of zeros in the disk of radius r centered at the origin have the same distribution.

Corollary 1. Let $h_r = 4\pi r^2/(1-r^2)$ denote the hyperbolic area of $B(0, r)$. Then

$$\mathbb{P}(N_r = 0) = e^{-h_r \frac{\pi}{24} + o(h_r)}.$$

These results generalize to other simply connected domains with smooth boundary. The idea is to consider conformal images of the functions z^n on the disk, which form a basis f_n of $H^2(\Omega)$, the set of analytic functions in Ω in $L^2(\partial\Omega)$.

Conjecture 1. For any function h analytic in $L^2(U(0,1))$ which vanishes at z_j for every j is the zero function.

For all the Gaussian analytic functions, there is a natural dynamics. Let

$$f_U(t, z) = \sum a_n(t) z^n,$$

where $a_n(t)$ is an Ornstein Uhlenbeck diffusion, i.e.,

$$a_n(t) = e^{-t/2} W_n(e^t).$$

Suppose that the zero set of f_U contains the origin. Movement of this zero locally satisfies the SDE

$$dz = \sigma dW,$$

where the diffusion constant σ depends on the locations of all the other points. In fact, this dependence is rather explicit: one multiplies together the inverse distances to the nearby points (the effect of the others being minimized by a dampening factor).

Watching videos of these dynamics reveals something surprising. The repulsion is achieved by a time speed-up when points get near to one another, since they cannot use drift to achieve repulsion.

Question: Can you study non-Gaussian polynomials? For example, take a Poisson point process and form a suitable Weierstrass product? Answer: More work has been done on Dirichlet...

Question: Can you study non-Gaussian coefficients? Answer: Yes: the Kac-Rice formula.