The i.i.d. Gaussian power series

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Joint work with Bálint Virág

Zeros of the *i.i.d.* Gaussian power series [Virág-P.].

Let

$$\begin{aligned} f_U(z) &= \sum_{n=0}^{\infty} a_n z^n \\ Z_U &= \operatorname{zeros}(f_U) \end{aligned} \tag{1}$$

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Theorem (Hannay, Zelditch-Shiffman, ...)

Law of $\mathbf{Z}_U$ invariant under Möbius transformations $z \to e^{i\alpha}\frac{z-\lambda}{1-\bar{\lambda}z}$ that preserve unit disk.

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Thus,

$$\begin{aligned} &\mathrm{Cov}[f_{\mathbb{C}}(z+a), f_{\mathbb{C}}(w+a)] = e^{(z+a)(\bar{w}+\bar{a})} \\ &= \mathrm{Cov}\left[e^{|a|^2/2} e^{\bar{a}z} f_{\mathbb{C}}(z), e^{|a|^2/2} e^{\bar{a}w} f_{\mathbb{C}}(w)\right]. \end{aligned}$$

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Since Gaussian processes are determined by $\mathrm{Cov}(\cdot,\cdot)$ this proves translation invariance of Law[zeros($f_{\mathbb{C}}$)].

Definition

Let $p_\epsilon(z_1, \ldots, z_n)$ denote the probability that a random function f has zeros in $B_\epsilon(z_1), \ldots B_\epsilon(z_n)$. **Joint intensity** of zeros (if it exists) is defined to be

$$p(z_1, \ldots, z_n) = \lim_{\epsilon \downarrow 0} \frac{p_\epsilon(z_1, \ldots, z_n)}{(\pi \epsilon^2)^n} \tag{2}$$

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Theorem (Hammersley)

Let f be a Gaussian analytic function in a planar domain D, $z_1, \ldots, z_n \in D$, and consider the matrix $A = \left(\mathbf{E} f(z_i)\overline{f(z_j)}\right)$. If A is non-singular then $p(z_1, \ldots z_n)$ exists and equals

$$\frac{\mathbf{E}\left(|f'(z_1)\cdots f'(z_n)|^2 \,\middle|\, f(z_1)=\cdots=f(z_n)=0\right)}{\det(\pi A)}.$$

Theorem (Virág - P.)

The joint intensity of zeros for f_U is

$$\begin{aligned} p(z_1, \ldots, z_n) &= \pi^{-n} \det \left[\frac{1}{(1 - z_i \bar{z}_j)^2} \right]_{i,j} \\ &= \det[K(z_i, z_j)] \end{aligned}$$

where $K(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}$ is the Bergman kernel for U.

Theorem (Virág - P.)

Let

$$X_k \sim \begin{cases} 1 & r^{2k} \\ 0 & 1 - r^{2k} \end{cases}$$

be independent. Then $\sum_1^\infty X_k$ *and* $N_r = |\mathbf{Z}_U \cap B(0, r)|$ *have same distribution.*

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Corollary

Let $h_r = 4\pi r^2/(1 - r^2)$ *(hyperbolic area). Then*

$$\mathbf{P}(N_r = 0) = e^{-h_r \frac{\pi}{24} + o(h_r)} = e^{\frac{-\pi^2/12 + o(1)}{1-r}}.$$

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All of the above generalize to other simply connected domains with smooth boundary.

$$\mathbf{E}\left(f_D(z)\overline{f_D(w)}\right) = 2\pi S_D(z, w) \ \ (\text{Szëgo Kernel})$$

Denote $q = r^2$. Key to law of $N_r = |\mathbf{Z}_U \cap B(0,r)|$:

$$\begin{aligned}
\mathbf{E}\binom{N_r}{k} &= \frac{1}{k!}\int_{B_r^k} p(z_1,\ldots,z_k)dz_1,\ldots dz_k \\
&= \frac{q^{\binom{k+1}{2}}}{(1-q)(1-q^2)\ldots(1-q^k)} \\
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$$\sum_{k=0}^{\infty}\gamma_k s^k = \prod(1+q^k s),$$

implies that

$$\mathbf{E}(1+s)^{N_r} = \sum_{k=0}^{\infty}\mathbf{E}\binom{N_r}{k}s^k = \sum\gamma_k s^k$$

has product form!

Dynamics

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$$f_U(t, z) = \sum_n a_n(t) z^n$$

with $a_n(t)$ performing Ornstein-Uhlenbeck diffusion,

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Suppose that the zero set of f_U contains the origin. Movement of this zero satisfies stochastic differential equation

$$dz = \sigma dW$$

where

$$\frac{1}{\sigma} = |f_U'(0)| = c \lim_{r \uparrow 1} \frac{1}{\sqrt{1-r^2}} \prod_{\substack{z \in Z_U \\ 0<|z|<r}} |z| = \tilde{c} \prod_{k=1}^{\infty} e^{1/k} |z_k|. \qquad (3)$$