Random Metric and Conformal Geometries: some old and new questions
Francois David
MSRI workshop on conformal invariance and statistical mechanics
Lecture notes, 9:30 am, March 29, 2012
Notes taken by Samuel S Watson

Question 1. (Polyakov 1981) Can we define $\int D[g] \exp\left[-\mu_0 \int_M d^2z\sqrt{g}\right]$, where g is a two dimensional Riemann metric, and μ_0 is a cosmological constant.

If so, then we'll have a two-dimensional theory of quantum gravity. We fix the gauge

$$g = \hat{g}e^{\phi(z,\bar{z})},$$

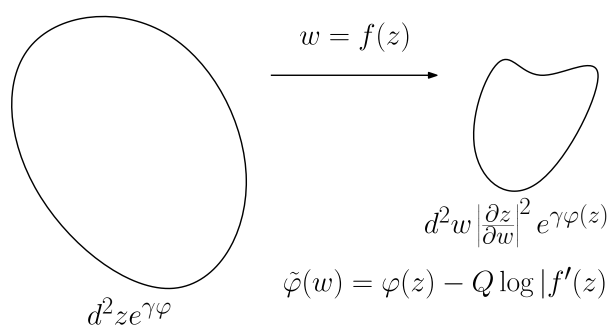
where φ is a conformal factor. Effective theory for $\varphi(z,\bar{z})$, conformal or Liouville field. An anomaly consistency condition and locality is

$$S_L(\varphi) = \frac{1}{2\pi}\int \sqrt{\hat{g}}\left[\frac{1}{2}(\hat{\nabla}\varphi)^2 + \frac{Q}{2}\hat{R}\varphi + \mu e^{\gamma\varphi},\right]$$

where $\mu = \mu_0 - x\Lambda^2$ is the normalized cosmological constant and $g = \hat{g}e^{\gamma\varphi}$. Note that the curvature is $R = -\mu < 0$. This is the *Liouville* theory. There is a relationship between Q and γ, namely

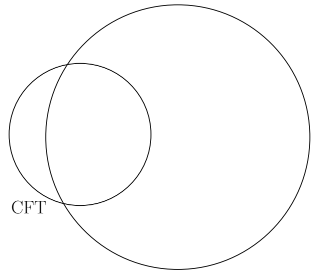
$$Q = \frac{\gamma}{2} + \frac{2}{\gamma}.$$

Conformal invariance of the field translates into a transformation property for φ.



Gravity and matter fields gives Liouville and matter CFT, with $c_L = 1 + 6Q^2$.

The KPZ relations can be developed from an algebraic CFT point of view, and also from a geometric point of view by Duplantier and Sheffield. The relationship between these perspectives is illustrated by the Venn diagram below. There are many things we learn from the geometric approach, but there are other important CFT questions to be addressed.



For example, we can ask about the renormalization of $\int D[g]$, which can be viewed as a random triangulation (random planar maps).

Is it possible to measure the fractal dimension of some X in the measure $e^{\gamma\varphi}$, where φ is the Gaussian free field. One approach is to cover with a diffusion process in the metric $e^{\gamma z}$ instead of with disks. More precisely, we run a diffusion process in the metric $e^{\gamma z}$ and ask how much of X is "covered" by the diffusion process. We can define $K(t) = e^{t\Delta}$, where Δ is the Laplace operator, which depends on φ by

$$\Delta = e^{-h\varphi(z,\bar{z})}\Delta_0 = e^{-h\varphi(z,\bar{z})}(-\partial_z\partial_{\bar{z}}).$$

One can compute (at least by CFT methods) the behavior of

$$\overbrace{d\mu_X^\circ(z,\bar{z})}^{\text{measure in flat space}} \quad \longrightarrow d\mu_x^Q(z,\bar{z}) = e^{-\gamma(1-x_\varphi)\varphi(z,\bar{z})}\, d\mu_x^0(z,\bar{z}).$$

The quantum heat kernel is

$$\int d\mu_x^Q(z,\bar{z})K^Q(t)(z,z_0) \simeq t^{-x_Q}.$$

Doing the calculation, we get

$$x_0 = x_Q + \frac{\gamma^2}{4}(x)(x-1).$$

This suggests that properties of random quadrangulations do in fact correspond to properties of Liouville quantum gravity.

There are many results available when $c = 0$. For example, work by Schaeffer, Miermont, Le Gall using the continuum random tree.

If $c \neq 0$, little is known. Numerical simulations are available for the case $c = 2$, while the there are conjectures about $(3+\sqrt{17})/2$ and 4.

It turns out that geodesics are strange objects in random geometries. Two important properties are

(a) Confluence: geodesics from A to C and from B to C coincide for a positive length (which is proved on a combinatorial level).

(b) Macroscopic uniqueness. There are sometimes multiple geodesics, but they are close to one another on the scale of the lattice mesh.

Question 2. Consider two points of Euclidean distance r from one another, and consider the geodesic connecting them in the quantum metric. What is its fractal dimension, and how does its quantum length depend on r?

It turns out that the dimension of the geodesic is $1 + \kappa/8$, which would suggest that the geodesic is an SLE path. However, the geodesic is correlated with h, so it is not an SLE path.

Old calculation: Perturbative calculation in Liouville theory, to first order in K. Consider a free massive scalar field ϕ, and consider

$$S[\phi] = \int d^2 z \frac{1}{2} (\partial\phi)^2,$$

under which

$$\langle \text{VAC} | \phi(x_A)\phi(x_B) | \text{VAC} \rangle \frac{m^2}{2}\phi^2 = \exp(-m \operatorname{dist}(x_A, x_B))$$

Now

$$\operatorname{dist}(x_A, x_B) = -\frac{1}{m} \log(\langle \phi(x_A)\phi(x_B) \rangle).$$

Computing this using Liouville theory

$$\langle \text{volume(ball of radius } R) \rangle = \pi R^2 + \kappa \sqrt{m} R^{5/2},$$

the second term being surprising.