Conformal invariance of spin correlations in the planar Ising model
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As we have seen, the planar Ising model is integrable. Rcall that the Hamiltonian for the Ising model is

$$H = -\sum_{i,j} \sigma_i \sigma_j,$$

where $\sigma_i \in \{\pm 1\}$ is the spin of the ith site, and the sum is over neighbors i and j. The critical parameter is $1/(\sqrt{2}+1)$, above which the system is disordered and below which it is highly ordered.

There are two points of view on conformal invariance. One on hand, we may consider interfaces or the whole ensemble of interfaces. On the other hand, we may consider the correlations between spins, or energy density, or fermonic observables.

$$\langle \sigma(z) \rangle_+^\Omega = \delta^{-1/8} \mathbb{E}_+^{\Omega^\delta} \sigma(z^\delta).$$

How can we obtain this exponent 1/8 with bare hands?

We define the basic fermionic observable:

$$F^\delta(z) := \frac{Z_{\text{config}:a\to z} e^{-\frac{i}{2}\theta(a\to z)}}{Z_{\text{config}:a\to z} e^{-\frac{i}{2}\theta(a\to b)}}.$$

The function F^δ is discrete holomorphic, i.e., satisfies some discrete version of the Cauchy-Riemann identities. This may be proved by a simple combinatorial bijection between configurations.

Theorem 1. As $\delta \to 0$, properly normalized at the point b, discrete holomorphic observables $\delta^{-1/2} F^\delta$ converge to a square root of a derivative of a certain conformal map.

We may remove an edge a and define study the correlation of spins associated with edges adjacent to this one. It turns out that this converges to a conformally invariant quantity as well.

Theorem 2. As $\delta \to 0$, the ratio $\mathbb{E}_{ab}[\sigma(z^\delta)]/\mathbb{E}_+[\sigma(z^\delta)]$ tends to the conformally invariant limit $\cos[\pi \operatorname{hm}_\Omega(z, (ba))]$.

Explicit formulas for $\langle \sigma_{a_0} \sigma_{a_1} \ldots \sigma_{a_n} \rangle_\Omega^+$ were predicted by CFT methods (John Cardy). These formula (for two points, say) depend on the hyperbolic distance from a to b in Ω.

When $k \geq 2$, the answer depends on the quantity

$$\mathcal{L}_\Omega(a_0, \ldots, a_k) = \sum \operatorname{Re}[\mathcal{A}_\Omega(a_j; a_0, \ldots, \hat{a}_j, a_k)\, da_j],$$

with coefficients $\mathcal{A}_\Omega$ given explicitly (see slides).

Theorem 3.

$$\frac{\mathbb{E}^{\text{free}}_{\Omega_\delta}[\sigma_{a+\delta}\sigma_{b+\delta}]}{\mathbb{E}^{+}_{\Omega_\delta}\sigma_a\sigma_b}$$

converges to a conformally invariant limit $\exp(-\frac{1}{2}d^{\text{hyp}}_{\Omega}(a,b))$.

The main tool is an observable branching at the source $a \in \Omega$. We use a double cover branched at a point.