Conformal blocks in 2d CFT, the Calogero-Sutherland model and the AGT conjecture

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Conformal blocks of a 2d CFT

$$c = 1 - 6\frac{(g-1)^2}{g} \qquad\qquad g = \frac{\kappa}{4}$$

Fractional Quantum Hall Effect
(anyons) with non-abelian statistics

$$g \leftrightarrow -g$$

AGT conjecture
Nekrasov's partition function ~
Liouville conformal blocks

Integrable structure of the Calogero-Sutherland model

[Alday, Gaiotto, Tachikawa, 09]

AGT conjecture

Link between a supersymetric 4d theory and a 2d CFT
(via an integrable model)

n-point correlation function of Liouville theory $\longleftrightarrow$ Nekrasov's instanton partition function of a gauge theory with gauge group $su(2)_1 \otimes \ldots \otimes su(2)_{n-3}$

$$Z^{u(2)} = Z^{u(1)} Z^{su(2)}$$

$$\langle V_{\Delta_{-1}}(\infty)\, V_{\Delta_0}(1)\, V_{\Delta_1}(q_1) \ldots V_{\Delta_{n-3}}(q_1 \ldots q_{n-3})\, V_{\Delta_{n-2}}(0)\rangle = c \prod_i f(\Delta_i) \int \prod_i a_i^2 da_i\, |Z(q|\Delta, \widetilde{\Delta})|^2$$

[Nekrasov, 02]

$$\alpha_i = Q/2 + a_i$$

$$Q = b + \frac{1}{b}$$

$$\Delta_i = m_i(Q - m_i), \quad \widetilde{\Delta}_i = \alpha_i(Q - \alpha_i)$$

AGT dictionary:

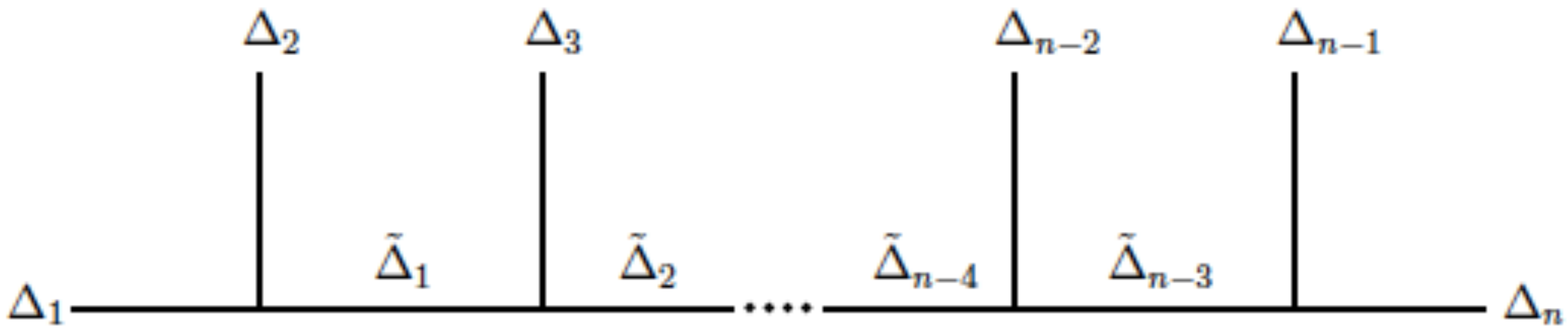
[Alday, Gaiotto, Tachikawa, 09]

Gauge theory	Liouville theory
Deformation parameters ϵ_1, ϵ_2	Liouville parameters $\epsilon_1 : \epsilon_2 = b : 1/b$ $c = 1 + 6Q^2$, $Q = b + 1/b$
four free hypermultiplets	a three-punctured sphere
Mass parameter m associated to an $SU(2)$ flavor	Insertion of a Liouville exponential $e^{2m\phi}$
one $SU(2)$ gauge group with UV coupling τ	a thin neck (or channel) with sewing parameter $q = \exp(2\pi i \tau)$
Vacuum expectation value a of an $SU(2)$ gauge group	Primary $e^{2\alpha\phi}$ for the channel, $\alpha = Q/2 + a$
Instanton part of Z	Conformal blocks
One-loop part of Z	Product of DOZZ factors
Integral of $\lvert Z^2_{\text{full}} \rvert$	Liouville correlator

$$\langle V_{\Delta_{-1}}(\infty)\, V_{\Delta_0}(1)\, V_{\Delta_1}(q_1) \ldots V_{\Delta_{n-3}}(q_1 \ldots q_{n-3})\, V_{\Delta_{n-2}}(0) \rangle = c \prod_i f(\Delta_i) \int \prod_i a_i^2 da_i \, |Z(q|\Delta, \widetilde{\Delta})|^2$$

CFT side: computing the conformal blocks

$$\langle V_{\Delta_{-1}}(\infty)\, V_{\Delta_0}(1)\, V_{\Delta_1}(q_1) \dots V_{\Delta_{n-3}}(q_1 \dots q_{n-3})\, V_{\Delta_{n-2}}(0)\rangle = c \prod_i f(\Delta_i) \int \prod_i a_i^2 da_i\, |Z(q|\Delta, \widetilde{\Delta})|^2$$



[Belavin, Polyakov, Zamolodchikov, 84]

- insert complete set of states $\sum_{\mu_i} |\mu_i\rangle\langle\mu_i|$ in the intermediate channels

$$|\mu_i\rangle \sim L_{-n_1} \dots L_{-n_k} |\widetilde{\Delta}_i\rangle$$

- compute the matrix elements elements $\dfrac{\langle \nu_i | V_{\Delta_{i-1}}(1) | \mu_{i+1}\rangle}{\langle \nu_i | \mu_{i+1}\rangle}$

- compare with the gauge theory result **[Nekrasov, 02]**

$\longrightarrow$ (physicist's) proof of the AGT conjecture **[Alba, Fateev, Litvinov, Tarnoplosky, 10]**

Proof of AGT in mathematical literature:

general proof : **[Maulik, Okounkov, unpublished]**

u(1) case : **[Carlsson, Okounkov, 08]**

W_k case : **[Schiffmann, Vasserot, 12]**

also:

q-deformed analog (Ding-Iohara algebra) :

[Awata, Feigin, Hoshino, Kanai, Shiraishi, Yanagida, 11]

Overview

- Review of the Calogero-Sutherland model
- CS model and the conformal blocks of Virasoro theories
- Bosonisation and CS integrals of motion
- W_k symmetry

2d CFT and the Calogero-Sutherland model

Trigonometric CS model: set of N commuting Hamiltonians for N particles on a circle:

$$H_1^g = \mathcal{P} = \sum_{i=1}^{N} z_i \partial_i$$

$$H_2^g = H^g = \sum_{i=1}^{N} (z_i \partial_i)^2 - g(g-1) \sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2} \qquad z_j = e^{2i\pi x_j / L}$$

$$H_3^g = \sum_{i=1}^{N} (z_i \partial_i)^3 + \frac{3}{2} g(1-g) \sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2} (z_i \partial_i - z_j \partial_j).$$

...

Two different **boundary conditions** for the wave functions:

$$\Psi^+(z) = \Delta^g(z) F^+(z) \qquad \text{or} \qquad \Psi^-(z) = \Delta^{1-g}(z) F^-(z)$$

$$\Delta^\gamma(z) = \prod_{i<j} (z_i - z_j)^\gamma$$

2d CFT and the Calogero-Sutherland model

Eigenfunctions with abelian monodromy:

$$\Psi^+_\lambda(z) = \Delta^g(z) J^{1/g}_\lambda(z) \qquad\qquad \Psi^-_\lambda(z) = \Delta^{1-g}(z) J^{1/(1-g)}_\lambda(z)$$

Jack polynomials: eigenfunctions of the Hamiltonian $\qquad \alpha^{-1} = g \quad \text{or} \quad 1-g$

$$\mathcal{H}^\alpha = \sum_{i=1}^N (z_i \partial_i)^2 + \frac{1}{\alpha} \sum_{i<j}^N \frac{z_i + z_j}{z_{ij}} (z_i \partial_i - z_j \partial_j)$$

$$\mathcal{E}^\alpha_\lambda = \sum_i^N \lambda_i \left[\lambda_i + \frac{1}{\alpha}(N + 1 - 2i) \right]$$

characterized by **partitions** λ with λ_i integers: $\qquad \lambda_1 \geq ... \geq \lambda_N \geq 0$

2d CFT and the Fractional Quantum Hall Effect

holomorphic correlators $\longleftrightarrow$ wave functions for the FQHE

free boson (c=1): $\langle \phi(z)\phi(w)\rangle = -\ln(z-w)$

$$J(z) = i\partial_z \varphi(z) = \sum_{n\in\mathbf{Z}} \frac{a_n}{z^n} \qquad [a_n, a_m] = n\delta_{n+m,0}$$

primary fields: vertex operators $V_\beta =: e^{i\beta\phi(z)}:$

"electron" operator:

$$V_e(z) =: e^{i\sqrt{m}\phi(z)}:$$

"quasi-hole" operator:

$$V_q(w) =: e^{i/\sqrt{m}\phi(w)}:$$

$$\langle V_e(z_1)\ldots V_e(z_N)\, V_q(w_1)\ldots V_q(w_M)\rangle \sim \prod_{i<j} z_{ij}^m \prod_{i<j} w_{ij}^{1/m} \prod_{i,j}(z_i - w_j)$$

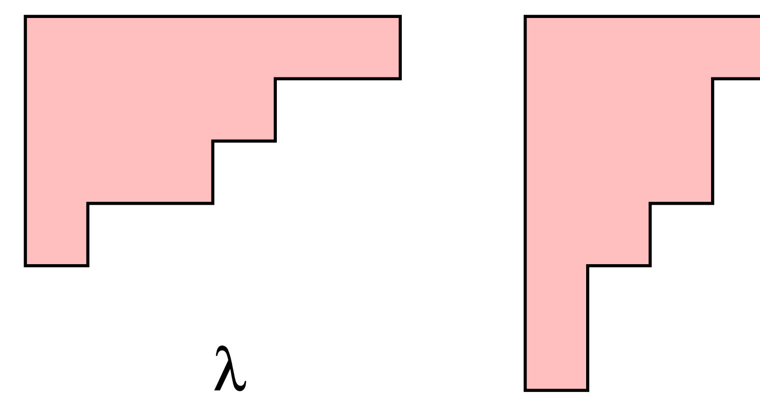
2d CFT and the Calogero-Sutherland model

Duality $g \to 1/g$: [Stanley 89; Macdonald 88; Gaudin 92]

$$\left[\mathcal{H}^{1/g} + g\,\mathcal{H}^{g} + C(N,M)\right] \prod_{i=1}^{N}\prod_{i=1}^{M}(1+z_i w_j) = 0$$

$$\prod_{i,j}(1+z_i w_j) = \sum_{\lambda} J_{\lambda}^{1/g}(z) J_{\lambda'}^{g}(w)$$

Dual partitions:



λ

λ’

Apply this property to the state:

$$\langle V_e(z_1)\ldots V_e(z_N)\, V_q(w_1)\ldots V_q(w_M)\rangle \sim \prod_{i<j} z_{ij}^{m} \prod_{i<j} w_{ij}^{1/m} \prod_{i,j}(z_i - w_j)$$

States with non-abelian monodromy and Virasoro theories

Virasoro models with central charge : $$c = 1 - 6\frac{(g-1)^2}{g}$$ Ising: $g = 4/3$

Degenerate field with dimensions : $$\Delta_{(r|s)} = \frac{1}{4}\left(\frac{r^2-1}{g} + (s^2-1)g + 2(1-rs)\right)$$

Two second-level degenerate fields : $$\left(L_{-1}^2 - gL_{-2}\right)\Phi_{(1|2)} = 0\,, \qquad \left(L_{-1}^2 - \frac{1}{g}L_{-2}\right)\Phi_{(2|1)} = 0$$

When inserted in correlation function, the null-vector conditions translate into differential equations:

$$\mathcal{O}^g(z)\langle\Phi_{(1|2)}(z)\Phi_{\Delta_1}(z_1)\ldots\Phi_{\Delta_N}(z_N)\rangle = 0$$

with

$$\mathcal{O}^g(z) = \frac{\partial^2}{\partial z^2} - g\left(\sum_{j=1}^{N}\frac{\Delta_i}{(z-z_j)^2} + \frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right)$$

States with non-abelian monodromy and Virasoro theories

The correlators of second order degenerate fields

$\langle \Phi_{(2|1)}(w_1) \cdots \Phi_{(2|1)}(w_M) \rangle^a$ and $\langle \Phi_{(1|2)}(z_1) \cdots \Phi_{(1|2)}(z_N) \rangle^b$

are groundsates of the CS model with non-abelian monodromy **[Cardy, 04]**

They can be represented as Coulomb integrals **[Dotsenko, Fateev, 84]**

$$\langle \prod_i^N \Phi_{\alpha_i}(z_i) \rangle \propto \prod_i^n \prod_j^m \oint_{C_i} dx \oint_{S_j} dy \quad \langle \prod_i e^{i\alpha_i \phi(z_i)} \prod_j^n V_+(x_j) \prod_j^m V_-(y_j) \rangle$$

$V_+(x_j)$ $V_-(y_j)$ screening operators with charges $\alpha_+ = \sqrt{\frac{2}{g}}\,, \quad \alpha_- = -\sqrt{2g}\,,$

$$2\alpha_0 = \alpha_+ + \alpha_-$$

neutrality condition: $\sum_i \alpha_i + n\alpha_+ + m\alpha_- = 2\alpha_0$

States with non-abelian monodromy and Virasoro theories

$$\langle \Phi_{(2|1)}(w_1)\cdots\Phi_{(2|1)}(w_M)\rangle^a \qquad \langle \Phi_{(1|2)}(z_1)\cdots\Phi_{(1|2)}(z_N)\rangle^b$$

$$h = \frac{3g}{4} - \frac{1}{2}$$
$$\widetilde{h} = \frac{3}{4g} - \frac{1}{2}$$

How to characterize the excited states above these ground states?

consider the mixed correlators dressed by an extra u(1) component
[Bernevig, Estienne, Santachiara, 10]

$$\mathcal{F}^{a,b}_{M,N}(w;z) \equiv \langle \Phi_{(2|1)}(w_1)\cdots\Phi_{(2|1)}(w_M)\Phi_{(1|2)}(z_1)\cdots\Phi_{(1|2)}(z_N)\rangle_{a,b} \prod_{1\le i<j}^{M} w_{ij}^{2\widetilde{h}} \prod_{i,j}(w_i - z_j)^{1/2} \prod_{1\le i<j}^{N} z_{ij}^{2h}$$

Duality :

$$\left[h^{\alpha}(z) + g\, h^{\widetilde{\alpha}}(w)\right] \mathcal{F}^{a,b}_{M,N}(w;z) = 0$$

$$h^{\alpha}(z) \equiv \mathcal{H}^{\alpha}(z) - \mathcal{E}_0^{\alpha} + \left(\frac{N-2}{\alpha} - 1\right)[\mathcal{P}(z) - \mathcal{P}_0] - \frac{NM(M-2)}{4}$$

$$\alpha^{-1} = 1 - g\,, \qquad \widetilde{\alpha}^{-1} = 1 - g^{-1}$$

Calogero-Sutherland

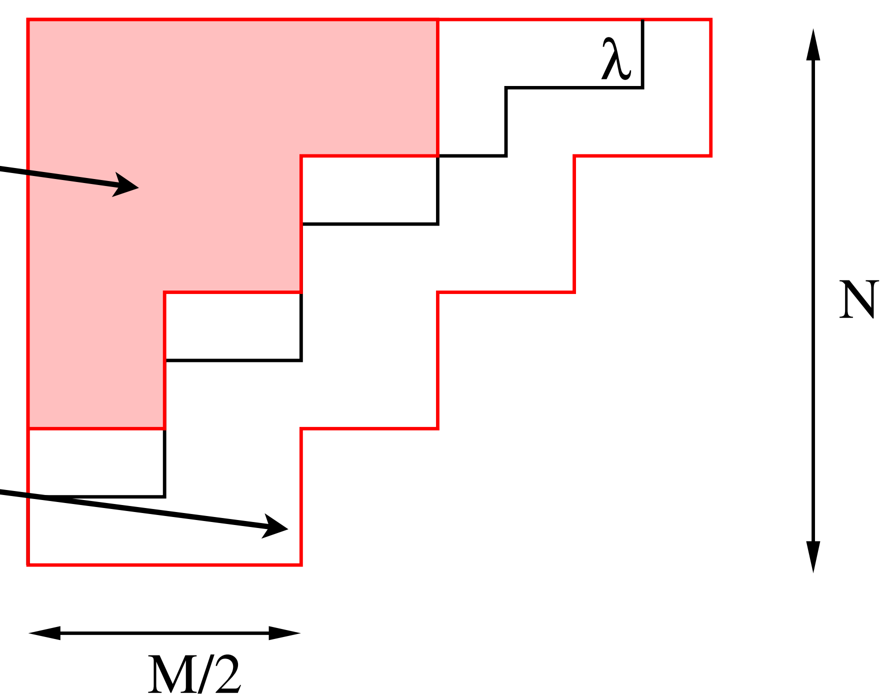
States with non-abelian monodromy and Virasoro theories

Consequence of duality: $$\mathcal{F}^{a,b}_{M,N}(w;z) = \sum_{\lambda} P^{\tilde{\alpha},a}_{\lambda'}(w)\, P^{\alpha,b}_{\lambda}(z)$$

$$\mathcal{E}^{\alpha}_{\lambda} = \sum_{i}^{N} \lambda_i \left[\lambda_i + \frac{1}{\alpha}(N+1-2i)\right] \qquad \alpha^{-1} = 1-g\ , \qquad \tilde{\alpha}^{-1} = 1-g^{-1}$$

smallest "partition" $\lambda^0_{2i-1} = \lambda^0_{2i} \;=\; 2h(N-2i)$

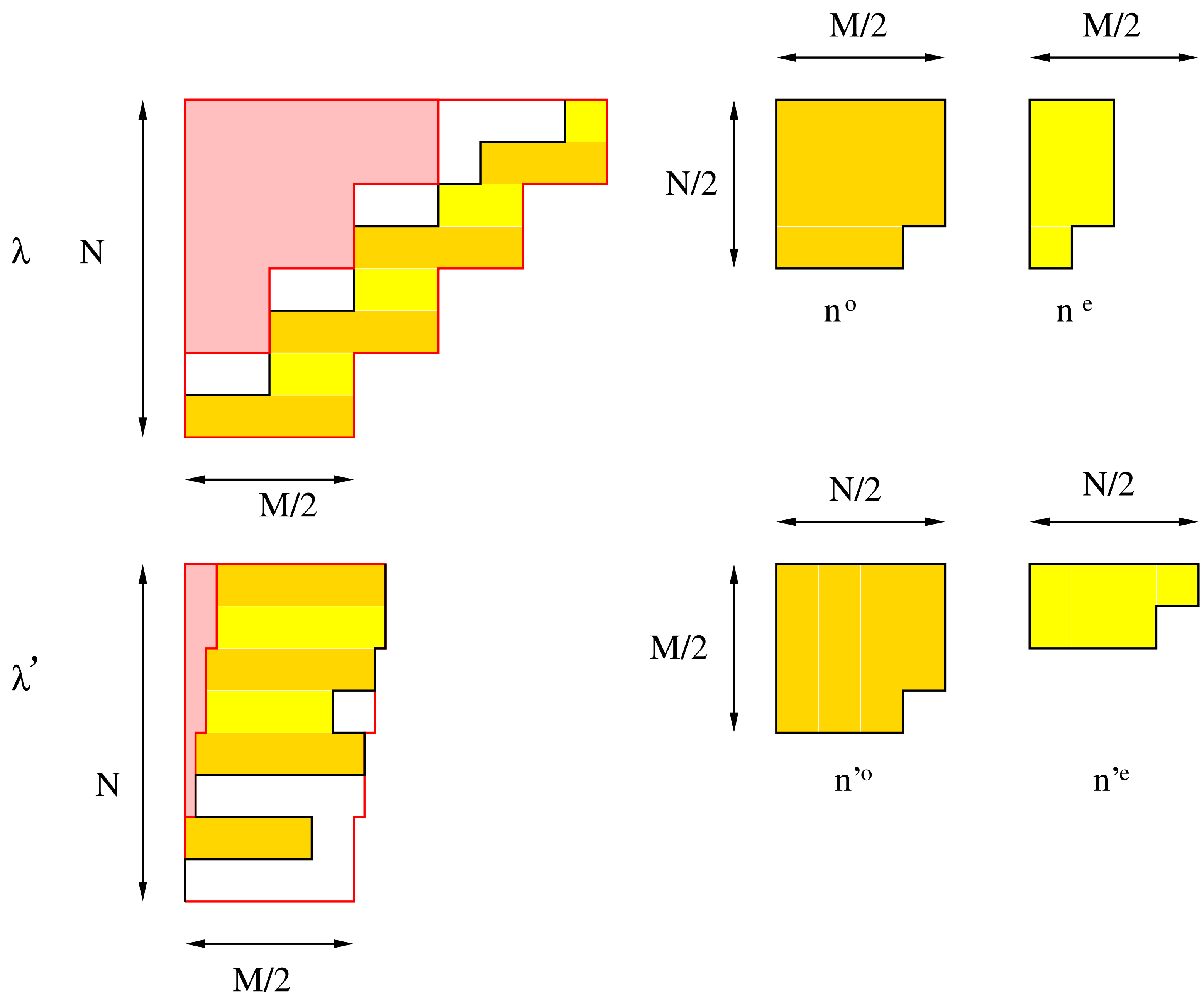
largest "partition" $\Lambda^0_i \;=\; \lambda^0_i + \frac{M}{2}$



2h is not integer in general

$\Rightarrow \qquad \lambda_i$ not integers

An excited state λ is characterized by **two partitions** n_e and n_o (reminiscent of AGT conjecture):



$$b(n) \equiv 2\sum_i (i-1)n_i = \sum_i n'_j(n'_j - 1)$$

$$|n| \equiv \sum_i n_i = |n'|$$

$$\mathcal{E}_\lambda^\alpha = [b(n'^o) + b(n'^e)] - g\,[b(n^o) + b(n^e)] + ((1-g)N - M + g)(|n^o| + |n^e|) + 2(g-1)|n^e| + \mathcal{E}^\alpha_{(M/2)^N}$$

Bosonized version of the CS Hamiltonian

Heisenberg algebra : $[a_n, a_m] = n\delta_{n+m,0}$

Virasoro algebra : $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}.$

Feigin-Fuchs representation : $L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} : b_{n-m}b_m : -\alpha_0(n+1)b_n$

$$2\alpha_0 = \sqrt{\frac{2}{g}} - \sqrt{2g}$$

basis for the Hilbert space:

$$a_{-m_1}\dots a_{m_l}L_{-n_1}\dots L_{-n_k}|q\rangle$$

Degenerate fields dressed by u(1) vertex operators :

$$V(z) \equiv \Phi_{12}(z)\, e^{i\sqrt{\frac{g}{2}}\phi(z)}\ , \qquad \widetilde{V}(w) \equiv \Phi_{21}(w)\, e^{i\frac{1}{\sqrt{2g}}\phi(w)}$$

Bosonized version of the CS Hamiltonian

Consider generic correlation functions :

$$f^+_\mu(z_1, z_2, \cdots, z_N) = \langle \mu | V(z_1) V(z_2) \cdots V(z_N) | P \rangle$$
$$f^-_\mu(z_1, z_2, \cdots, z_N) = \langle P | V(z_1) V(z_2) \cdots V(z_N) | \mu \rangle$$

$|\mu\rangle$: generic state (primary or descendant)

$|P\rangle$: primary state

Translate the CS action on states :

$$H^g_n f^\pm_\mu(z_1, z_2, \cdots, z_N) = \sum_\nu \left[I^\pm_{n+1}(g) \right]_{\mu,\nu} f^\pm_\nu(z_1, z_2, \cdots, z_N)$$

$$I^{(\pm)}_3(g) = 2(1-g) \sum_{m\geq 1} m a_{-m} a_m \pm \sqrt{2g} \sum_{m \neq 0} a_{-m} L_m \pm \sqrt{\frac{g}{2}} \left(\sum_{m,k\geq 1} a_{-m-k} a_m a_k + a_{-m} a_{-k} a_{m+k} \right)$$

$$I^{(\pm)}_n(g) \propto I^{(\mp)}_n(1/g)$$

[Alba, Fateev, Litvinov, Tarnopolsky,10]
for Liouville $\quad g \to -g$

Bosonized version of the CS Hamiltonian

- rotate the boson basis: $$c_m = \frac{1}{\sqrt{2}}(a_m + b_m) \ , \qquad \widetilde{c}_m = \frac{1}{\sqrt{2}}(a_m - b_m)$$

[AFLT, 10; Belavin and Belavin, 11]

- introduce the one-component bosonised CS Hamiltonians:

$$\mathcal{I}_3^{\pm}(c;g) = (1-g)\sum_{m>0} m c_{-m} c_m \pm \sqrt{g}\sum_{m,k>0}(c_{-m-k}c_m c_k + c_{-m}c_{-k}c_{m+k})$$

[Jevicki, 91]
[Awata, Matsuo, Odake, Shiraishi, 95]

- related to the Benjamin-Ono hierarchy, *e.g.* [Abanov, Bettelheim, Wiegmann, 09]

Then the integrals of motion in Virasoro x u(1) have the triangular structure:

$$I_3^+(g) = \mathcal{I}_3^+(c;g) + \mathcal{I}_3^+(\widetilde{c};g) + \sqrt{2g}\,(b_0 - \alpha_0)(\mathcal{I}_2(c) - \mathcal{I}_2(\widetilde{c})) + 2(1-g)\sum_{m>0} m c_{-m}\widetilde{c}_m$$

also

[Maulik, Okounkov, unpublished]

[Shou, Wu, Yu, 11]

[Schiffmann, Vasserot, 12]

Constructing the eigenstates of the CS Hamiltonian

At g=1 the eigensstates can be constructed with the help of Schur polynomials
+ reflection condition

$$|n^o, n^e; q\rangle = |n^e, n^o; -q\rangle$$

$$c_{-n} \sim p_n = \sum_i x_i^n$$

$$|n^o, n'^e; q\rangle = S_{n^o}(c) S_{n^e}(\widetilde{c})|q\rangle + S_{n^o}(\widetilde{c}) S_{n^e}(c)| - q\rangle \qquad b_0|q\rangle = q|q\rangle$$

At arbitrary g, use the Jack basis:

$$|n^o, n^e; q\rangle = J_{n^o}^{1/g}(c)\ J_{n^e}^{1/g}(\widetilde{c})\ |q\rangle + \ldots$$

$|n^o, n^e; q\rangle$ is the basis used by AFLT to compute the matrix elements $\dfrac{\langle \nu_i | V_{\Delta_{i-1}}(1) | \mu_{i+1}\rangle}{\langle \nu_i | \mu_{i+1}\rangle}$

and prove the AGT conjecture

WA_{k-1} theories

[Fateev, Zamolodchikov, 85]

[Fateev, Lukyanov, 87]

The same construction extends to WA_{k-1} algebras (differential equations, duality, bosonisation)

k-1 bosons x u(1) component: $\longrightarrow$ k- component CS Hamiltonian

$$I_3^{(\pm)}(g) = 2(1-g)\sum_{m\geq 1} m a_{-m} a_m \pm \sqrt{2g}\sum_{m\geq 1}(a_{-m}L_m + L_{-m}a_m) \pm \sqrt{\frac{g}{2}}\left(\sum_{m,k\geq 1} a_{-m-k}a_m a_k + a_{-m}a_{-k}a_{m+k}\right) \pm \widetilde{W}_0$$

after basis rotation in the space of bosons:

$$I_3^{+}(g) = \sum_{j=1}^{k} \mathcal{I}^{\pm}(c^j; g) + 2(1-g)\sum_{j<l}\sum_{m\geq 1} m : c^j_{-m} c^l_m : + \quad \ldots$$

$\longrightarrow$ additive spectrum depending on **k partitions** (~ AGT conjecture for U(k) theories)

[see also Fateev, Litvinov, arXiv1109.4042]

Conclusions and remarks

- The states of the Vir x H CFT, or WA_{k-1} x H can be characterized in terms of eigenstates of the CS integrals of motion
- The CS integrable structure lies behind the AGT relation
- Extension to a large number of cases (*e.g*. coset and superconformal theories **[Belavin, Bershtein, Feigin, Litvinov, Tarnopolsky, 11]**; parafermionic theories **[Alfimov, Tarnopolsky, 11]**)

Open problems

- Theory of non-polynomial CS eigenfunctions?
- How to systematically generate the integrals of motion (transfer matrix, Q operators) in CFT?
- Relation with the integrable structure uncovered by **[Bazhanov, Lukyanov, Zamolodchikov, 94-98]**