The topology of the support of the equilibrium measure in the random normal matrix model

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Notes taken by Samuel S Watson

The random normal model is a probability measure on $n \times n$ normal matrices corresponding to

$$Z_n = \int e^{-2n\,\mathrm{Trace} Q(M)}\, dM.$$

The eigenvalues λ_j are interpreted as electrons in an external field nQ. The empirical law of these eigenvalues converges to a limiting measure (Johansson).

One may superimpose two copies of such an ensemble of eigenvalues and come up with an analogue of the Gaussian free field, and we can study its level curves. We will not discuss these curves, however, but rather deterministic ones of the limiting shape.

If the boundary has no singularities and no change in topology on (t_1, t_2), then ∂K_t is a classical solution of the HS equation.

For example, if $h(z) = a$, we get circles. For other choices of h, we can get fractal sets such as the Mandelbrot set, as well as cardioids and deltoids.

We say that K is a local droplet of Q if there is a neighborhood U of K such that K is a droplet of $Q + \infty \cdot 1_{U^c}$

Theorem 1. WZ hypotrochoids (including the deltoid) are the only local droplets in the case $h(z) = z^2$.

Theorem 2. Let $Q(z) = |z|^2 - H(z)$ and let $h = \partial H$ be a rational function of degree $d \geq 2$.

(i) The number of components in each local droplet set is $N \leq 2d - 2$.

(ii) If h is a polynomial, then $N \leq d - 1$.

(iii) These bounds are sharp.

Examples. If we have n disks inside a disk, we have $d = n$. If we have n discs inside an ellipse, we have $n = d + 1$. If we have n limaçons inside an ellipse, $d = 2n + 1$. In these three cases, the number of components of the interior of K is bounded above by $2n - 2$, $2n$, and $3n$, respectively. If we have n cardioids in an ellipse, $d = 2n + 1$.

Remark: The sharpness part of the results is not obvious.

Let Ω be a quadrature domain, and let r have degree d. Let $S : \Omega \to \hat{\mathbb{C}}$ be an S-function, agreeing with $\bar{z}$ on the boundary of Ω.