- Speaker:Alejandro Ramirez
- Title: Criteria for ballistic behavior of random walks in random environment
- Note taker:Xiaoqin Guo

0.
   - Definition and overview
   - Renormalization methods: weaken ballisticity conditions
   - Atypical quenched exit estimates (more powerful renormalization methods)

1. Notation:
   $U = \{e \in \mathbb{Z}^d : |e| = 1\}$ is the set of unit vectors.
   $\mathcal{P} = \{(p(e))_{e \in U} : p(e) \geq 0, \sum p(e) = 1\}$: jump probabilities
   $\Omega = \mathcal{P}^{\mathbb{Z}^d}$: the environment space.
   $\omega = \{\omega(x \in \mathcal{P} : x \in \mathbb{Z}^d\} \in \Omega$: a typical environment.
   X_n: random walks on lattice $\mathbb{Z}^d$.
   $P_{x,\omega}$: transition probability, $P_{x,\omega}(X_{n+1} = x + e | X_n = x) = \omega(x, e)$.
   μ: the law of the environment.
   $P_x := \int P_{x,\omega}\, \mathrm{d}\mu$ is the annealed law.
   Assume:
   1) $\{\omega(x) : x \in \mathbb{Z}^d\}$ is iid
   2) uniform ellipticity, ie, $\omega(x) \geq \kappa$ for positive constant κ, μ-a.s.

2. Transience and ballisticity: for $\ell \in S^{d-1}$, we say the RWRE is *transient* in the direction ℓ if
   $$\lim_{n\to\infty} X_n \cdot \ell = \infty \quad P_0\text{-almost surely,}$$
   and *ballistic* in the direction ℓ if
   $$\varliminf_{n\to\infty} \frac{X_n \cdot \ell}{n} > 0 \quad P_0\text{-almost surely.}$$

3. Open problem: Is it true that for $d \geq 2$ and μ iid and uniformly elliptic,
   $$\text{transience in } \ell \Longrightarrow \text{ballisticity in } \ell?$$
   Remark: When $d = 1$, the conjecture is not true.
   Explain:
   1) For $d = 1$, we say that a box of length $c \log n$ is "bad" if
   $$P_{x,\omega}(\text{exit time from the box} > n) \geq 0.5.$$
   For $s < 1, \epsilon > 0$, let
   $$G_n = \{\#\text{bad boxes in}[0, n^s] \geq n^\epsilon\}.$$

It turns out that (if we choose μ carefully) $\mu(G_n) \to 1$ as $n \to 1$. Hence with high probability, the time the RW exits $[0, n^s]$ is more than $n \Longrightarrow$ no ballisticity!
2) For $d \geq 2$,the cost $e^{-c(\log n)^d}$ of a trap (ie, a bad box) of radius $c \log n$ is not big enough.

4.

$$\text{Transience in } \ell \Longrightarrow \lim_{L \to \infty} P_0(T_{-L} < T_L) = 0.$$

Sznitman's conditions quantify how fast the limit goes to 0.

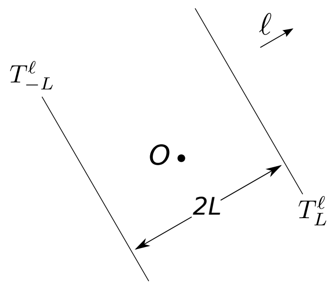


Figure 1: T_{-L} and T_L are exit times from the right and left sides of the slab.

Definitions:

- $(T)_\gamma|_\ell$: We say that condition $(T)_\gamma$ with respect to ℓ is satisfied if for large L,

$$P_0(T^{\ell'}_{-L} < T^{\ell'}_L) \leq \exp(-cL^\gamma), \quad \gamma \in (0,1)$$

  for all ℓ' in a neighborhood of ℓ.
- $(T) := (T)_1$.
- $(T') := (T)_\gamma$ is satisfied for all $\gamma \in (0,1)$.

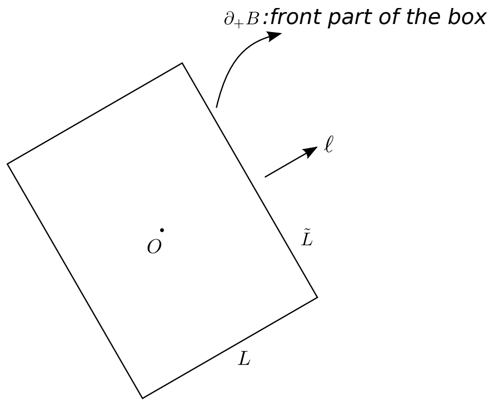
It is conjectured that for $d \geq 2$,

$$(T) \Leftrightarrow (T') \Leftrightarrow (T)_\gamma \Leftrightarrow \text{transience} \Leftrightarrow \text{ballisticity}.$$

History:

- Sznitman(2002):$(T') \Rightarrow$ ballisticity+LLN+annealed CLT
- Sznitman(2002): For $\gamma \in (0.5, 1)$, $(T)_\gamma \Leftrightarrow (T')$.
- Drewitz-R.(2012): $d \geq 4, \gamma \in (0,1)$, then $(T)_\gamma \Rightarrow (T')$.

5. Renormalization.



- Effective criterion (EC): Denote

$$\rho_{L,\tilde{L}} = \frac{P_{0,\omega}(\text{does not exit from } \partial_+ B)}{P_{0,\omega}(\text{exit } B \text{ from } \partial_+ B)}.$$

We say that the EC with respect to ℓ holds if

$$\inf_{a \in (0,1], L, \tilde{L}} L^{d-1} \tilde{L}^{3(d-1)+1} E_\mu[\rho^a_{L,\tilde{L}}] \leq 1. \tag{EC}$$

Sznitman (2002): EC$\Leftrightarrow$ (T').
We want to show: $(T)_\gamma \Rightarrow$EC. Strategy:

- Assuming $(T)_\gamma$, to get (EC), write

$$E_\mu[\rho^a_{L,\tilde{L}}] = A_0 + \sum_{j=1}^{n} A_j,$$

where

$$A_0 = E_\mu[\rho^a_{L,\tilde{L}}; P_{0,\omega}(\text{exit from } \partial B_+) \geq e^{-cL^\gamma}],$$
$$A_j = E_\mu[\rho^a_{L,\tilde{L}}; e^{-c_j L^{\beta_j}} \leq P_{0,\omega}(\text{exit from } \partial B_+) \leq e^{-c_{j-1} L^{\beta_{j-1}}}] \text{ for } 1 \leq j \leq n,$$

and $\gamma = \beta_0 < \beta_1 < \ldots < \beta_n$ is an increasing sequence with $\beta_n > 1$. (Note that $P_{0,\omega}(\text{exit from } \partial B_+)$ is never smaller than e^{-cL} due to uniform ellipticity.)

- Take

$$a = L^{-\alpha}, \quad 0 < \alpha < \gamma.$$

Suppose we know that for a square (ie, $L = \tilde{L}$) box B and $\beta \in (0,1)$,

$$\mu[P_{0,\omega}(\text{exit from } \partial B_+) \leq \exp(-L^\beta)] \leq \exp(-L^{f(\beta)})$$

for some nice function $f(\beta)$, then Jensen's inequality and $(T)_\gamma$ yield

$$A_0 \le e^{c_1 L^{\gamma-\alpha}} e^{-cL^{\gamma-\alpha}},$$
$$A_j \le e^{c_j L^{\beta_j-\alpha}} e^{-L^{f(\beta_{j-1})}}, \quad 1 \le j \le n.$$

We need $\alpha < \gamma$, $\beta_j < f(\beta_{j-1}) + \gamma$ for $j \le n$ and $1 < f(\beta_n) + \gamma$.

- Example: if we get $f(x) = x$, then $\beta_j \approx j\gamma$.

6. Atypical quenched exit estimate
Lemma For $\beta \in (\frac{1}{2}, 1)$,

$$f(\beta) = (\beta - 2) \wedge \frac{\gamma}{2} + (d-1)(\beta - \frac{1}{2}).$$

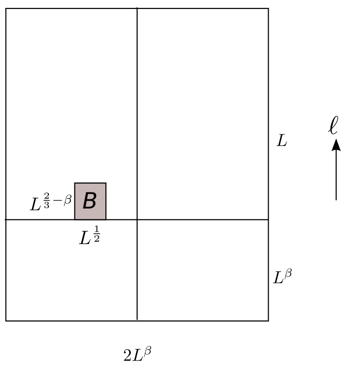


Figure 2:

Sketch: Divide a box as in the graphs into small blocks (of the same size as the box B in the graph). Call B "bad" if

$$P_{x,\omega}(\text{exit from } \partial_+ B) < \frac{1}{2}.$$

It turns out that $\mu(\text{a box is bad}) \le \exp(-L^{(\beta-\frac{1}{2})\wedge\frac{\gamma}{2}})$. Let

$$G = \{\#\text{bad boxes} \le L^{(d-1)(\beta-\frac{1}{2})}\}.$$

Computations show $\mu(G^c) \le e^{f(\beta)}$. Note that on the event G, we can find a "tube" of good boxes that connects the top and the bottom. This, together with uniform ellipticity yields

$$P_{x,\omega}(\text{exit from } \partial_+ B) \ge \kappa^{L^\beta} (\frac{1}{2})^{L^{\beta-\frac{1}{2}}} (\kappa^{L^{1/2}})^{L^{\beta-1/2}} \ge e^{-cL^\beta}. \quad \square$$

Corollary. For $\gamma > \frac{1}{3}$, $(T)_\gamma \Rightarrow (T')$.

7. Second renormalization method
Assume $(T)_{\gamma L}, \gamma(L) = \frac{c}{\ln \ln L}$. Divide the square box into small boxes of side L^ϵ (see Fig 3). Call B "good" if

$$\inf_{x \in \tilde{B}} P_{x,\omega}(\text{exit from } \partial_+ B) \geq 1 - e^{-L^\epsilon}.$$

Let

$$G = \{\#\text{bad boxes}\} \leq L^\beta.$$

On the event G, the probability that the walker exits the box from

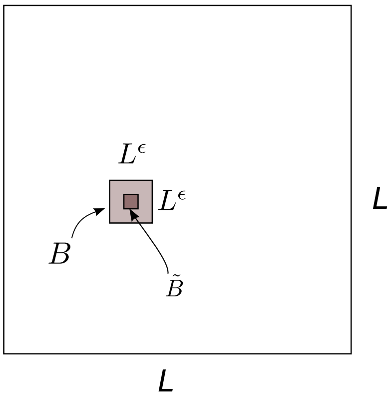


Figure 3: $\tilde{B}$ is the middle-third box in B.

the front is $\geq e^{L^{\beta+\epsilon}}$. Computations show that

$$\mu(G^c) \leq e^{-L^\beta}.$$

Corollary. For $\gamma(L) = \frac{c}{\ln \ln L}$, $(T)_\gamma \Rightarrow (T')$.

8. Criteria $(P)_M|_\ell$: for directions in a neighborhood of ℓ and L large,

$$P_0(T_{-L} - T_L) \leq \frac{1}{L^M}. \qquad ((P)_M)$$

Theorem(Berger,Drewitz,R.) $(P)_M \Rightarrow (T')$ for some $M = M(d)$.
Strategy: show that $(P)_M \Rightarrow (T)_{\gamma(L)}$, where $\gamma(L) = \frac{c}{\ln \ln L}$.