Edge Reinforced Random Walk, Vertex Reinforced Jump Process and a SuSy σ-model

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Edge Reinforced Random Walk (ERRW)

- $G = (V, E)$ a non-directed connected graph with finite degree at each site.
- $(a_e)_{e \in E}$, $a_e > 0$, positive weights on the edges.

The Edge Reinforced Random Walk (ERRW) on V starting at $i_0 \in V$, with initial weights (a_e), is the discrete time process (X_n) defined by $X_0 = i_0$ and

$$\mathbb{P}(X_{n+1} = j \,|\, X_k,\, k \leq n) = \mathbb{1}_{j \sim X_n} \frac{Z_n(\{X_n, j\})}{\sum_{i \sim X_n} Z_n(\{X_n, i\})}$$

where

$$Z_n(e) = a_e + \sum_{k=0}^{n-1} \mathbb{1}_{\{X_{k-1}, X_k\} = e}.$$

This model was introduced in 86 by Diaconis et Coppersmith.

- Diaconis, Freedman and Coppersmith, Diaconis : if G is finite, (X_n) is a mixture of reversible Markov chains with explicit distribution (the so-called "magic formula" or Coppersmith, Diaconis formula).
- Pemantle : strongly recurrent on $\mathbb{Z}$; a phase transition on trees. On a tree it is a RWRE (in Dirichlet environment).
- Merkl, Rolles : strong recurrence on a modification of $\mathbb{Z}^2$ for strong reinforcement.

Vertex Reinforced Jump Process (VRJP)

Let $(W_e)_{e\in E}$ be some conductances on the edges, $W_e > 0$.

The Vertex Reinforced Jump Process (VRJP) is the continuous time jump process $(Y_s)_{s\geq 0}$ defined by

$$\mathbb{P}(Y_{s+ds} = j \mid Y_u,\ u \leq s) = \mathbb{1}_{j\sim Y_s} W_{Y_s,j} L_j(s) ds,$$

with

$$L_i(s) = 1 + \int_0^s \mathbb{1}_{Y_u=i} du.$$

It means that, conditionally on the past at time s, if $Y_s = i$, Y jumps to $j \sim i$ with rate $W_{i,j} L_j(s)$.

- Model proposed by Werner
- Investigated on $\mathbb{Z}$ and on trees by Davis, Volkov, then Collevechio and Basdevant, Singh.

Results

- ERRW $\overset{\text{"loi"}}{=}$ VRJP with random independent conductances.
- The VRJP is a mixture of time-changed Markov jump processes with an explicit mixing law.
- This mixing law is a related to a supersymmetric σ-model introduced by Disertori, Spencer, Zirnbauer.

Using Disertori, Spencer, and Disertori, Spencer, Zirnbauer

- For the VRJP : strong recurrence in any dimension for strong reinforcement, i.e. W small (and in fact on graphs of bounded degree).
- For the VRJP : transience on $\mathbb{Z}^d$, $d \geq 3$, for small reinforcement, i.e. for W large.
- For the ERRW : a one line extension of Disertori, Spencer gives strong recurrence for the ERRW at strong reinforcement.

Afterwards, different proof of localization by Angel, Crawford, Kozma.

ERRW <—> VRJP : a time change

Time change of Y: let

$$A(s) = \sum_{i \in V} \log(L_i(s)),$$

$$X_t = Y_{A^{-1}(t)}.$$

Let $(T_i(t))$ be the local time of X.

$$T_i(t) = \int_0^t \mathbb{1}_{X_u = i} du$$

Conditionally on the past at time t, if $X_t = i$, X jumps to $j \sim i$ with rate

$$W_{i,j} e^{T_i(t) + T_j(t)}.$$

Indeed, if $t = A(s)$

$$e^{T_j(t)} = L_j(s), \;\; ds = e^{T_{X_t}(t)} dt.$$

ERRW <—> VRJP : Rubin and Kendall

Theorem (S., Tarres '11)

Let (W_e) be independent random conductances $W_e \sim Gamma(a_e)$. Let $(Y_{\tau_n})_{n\geq 0}$ be the discrete time process associated with the VRJP Y sarting at i_0. The annealed law of Y_{τ_n} is the law of the ERRW (X_n) starting at i_0 and with initial weights (a_e).

Three ingredients

- Rubin construction (Davis '90, then Sellke) : gives a continuous time version $\tilde{X}_t$ of the ERRW.
- Kendall transform : represent the Yule process as a mixture of time changed Poisson process.
- $\tilde{X}_t$ has the same law as the annealed law of the process (X_t) with independent random (W_e).

Mixing measure of the VRJP

$G = (V, E)$ **finite**. Let $N = |V|$. Fix the conductances (W_e).
Recall that, conditionally on the past at time t, if $X(t) = i$, X jumps to $j \sim i$ with rate

$$W_{i,j} e^{T_i(t) + T_j(t)},$$

where $(T_i(t))_{i \in V}$ is the local time of X. Let $\mathbb{P}_{i_0}$ be the law of (X_t) starting from $i_0 \in V$.

Proposition

For all $i_0 \in V$, and $i \in V$ the limit

$$U_i = \lim_{t \to \infty} T_i(t) - \frac{t}{N},$$

exists $\mathbb{P}_{i_0}$ a.s.

Heuristically: $X(t)$ accelerates with t ; at T "frozen", the uniform measure is a reversible measure for the process.

Theorem (S., Tarres '11)

i) Under $\mathbb{P}_{i_0}$, (U_i) *has distribution on* $\mathcal{H}_0 = \{(u_i), \sum u_i = 0\}$

$$\frac{1}{(2\pi)^{(N-1)/2}} e^{u_{i_0}} e^{-H(W,u)} \sqrt{D(W,u)},$$

with

$$H(W,u) = 2 \sum_{\{i,j\} \in E} W_{i,j} \sinh^2\left((u_i - u_j)/2\right)$$

and $D(W,u)$ *is any diagonal minor of the matrix* $M(W,u)$ *with size* $V \times V$ *and coefficients*

$$m_{i,j} = \begin{cases} W_{i,j} e^{u_i + u_j} & \text{if } i \neq j \\ -\sum_{k \in V} W_{i,k} e^{u_i + u_k} & \text{if } i = j \end{cases}$$

We denote by $\nu^{i_0,W}$ *this distribution.*

ii) Let $(U_i)_{i\in V}$ be the r.v. on $\mathcal{H}_0$ distributed under $\nu^{i_0,W}$. Let (Z_t) be the **Markovian** *jump process starting from i_0 with jump rates from i to j given by*

$$\frac{1}{2}W_{i,j}e^{U_j-U_i}.$$

Let $(l_i(t))_{i\in V}$ be the local time Z at time t. Let

$$B(t)=\sum_{i\in V}\sqrt{1+l_i(t)}-1,\quad \tilde{Y}_s=Z_{B^{-1}(s)}.$$

The annealed law of $\tilde{Y}$ is the law of the VRJP (Y_s) starting from i_0 (tacking expectation with respect to (U_i)).
In particular, the dicrete time process associated with (Y_s) is a mixture of reversible Markov chains with conductances $W_{i,j}e^{U_i+U_J}$.

Sketch of proof of i)

$(X_t, (T_i(t))_{i\in V})$ is a Markov process on $V \times \mathbb{R}^V$ with generator $\tilde{L}$

$$\tilde{L}f(i,T) = \frac{\partial}{\partial T_i} f + (L(T)f)(i,T),$$

where $L(T)$ is the generator of the Markov process at T "frozen"

$$L(T)g(i) = \sum_{j\sim i} W_{i,j} e^{T_i+T_j}(g(j)-g(i)).$$

We denote by $\mathbb{P}^W_{i,T}$ its law starting from the initial condition (i,T), and $\mathbb{P}^W_i = \mathbb{P}^W_{i,0}$. With

$$(W^T)_{i,j} = W_{i,j}e^{T_i+T_j},$$

clearly

$$\mathbb{P}^W_{i,T} = \mathbb{P}^{W^T}_i.$$

For $(\lambda_i) \in \mathcal{H}_0$

$$\begin{aligned}
\Psi(i_0, T, \lambda) &= \int_{\mathcal{H}_0} e^{i<\lambda,u>} \nu^{i_0, W^T}(du) \\
&= \frac{1}{\sqrt{2\pi}^{N-1}} \int_{\mathcal{H}_0} e^{i<\lambda,u>} e^{u_{i_0}} e^{-H(W^T,u)} \sqrt{D(W^T, u)} du.
\end{aligned}$$

we want to prove

$$\Psi(i_0, 0, \lambda) = \mathbb{E}_{i_0}^{W} \left(e^{i<\lambda,U>} \right).$$

Lemma

Ψ is solution of the Feynman-Kac equation

$$i\lambda_{i_0} \Psi(i_0, T, \lambda) + (\tilde{L}\Psi)(i_0, T, \lambda) = 0.$$

It follows that

$$\Psi(i_0, 0, \lambda) = \mathbb{E}_{i_0}^{W} \left(e^{i<\lambda, T(t)>} \Psi(X_t, T(t), \lambda) \right).$$

Clearly

$$e^{i<\lambda,T(t)>} = e^{i<\lambda,T(t)-t/N>} \to e^{i<\lambda,U>}.$$

We need to prove that

$$\lim_{t\to\infty} \Psi(X_t, T(t), \lambda) = 1.$$

It comes from the following fact : when t is large $W^{T(t)}$ is large and (u_i) is small. We can then make the following approximations

$$\begin{gathered} e^{ux_t} \sim 1 \\ 2\sinh^2((u_j - u_i)/2) \sim \tfrac{1}{2}(u_i - u_j)^2 \\ W_{i,j}^{T(t)} e^{u_i+u_j} \sim W_{i,j}^{T(t)} \\ e^{i<\lambda,u>} \sim 1 \end{gathered}$$

Integrating on the gaussian free field, the integral gives 1.

Relation with the SuSy sigma model

- The distribution $\nu^{i_0,W}$ is a marginal of a supersymmetric sigma model investigated by Disertori, Spencer, Zirnbauer : $(t, s, \overline{\psi}, \psi)$ where (t, s) are bosonic hyperbolic coordinates and $(\overline{\psi}, \psi)$ fermionic coordinates.
- This sigma model is related to Anderson localization, more precisely to gaussian random band matrices (Efetov, Wegner).
- A link between ERRW and this SuSy sigma model had been suspected by several mathematicians including Kozma, Heydenreich, Sznitman (but never made explicit).

Changing to coordinates $t_i = u_i - u_{i_0}$ the measure $\nu^{i_0,W}$ has density

$$e^{-\sum t_i} e^{-\sum_{\{i,j\}\in E} W_{i,j}(cosh(t_i - t_j) - 1)} \sqrt{D(W,t)}.$$

Corollaires

On $\mathbb{Z}^d$ with $W_{i,j} = \beta$, Disertori and Spencer proved that for any d and β small enough $\nu^{0,\beta}(e^{t_i/2})$ is exponentially decreasing.

Corollary (Disertori, Spencer 2010 + Theorème 2)

For any $d \geq 1$ there exists $\beta_c(d) > 0$ such that $\beta < \beta_c(d)$ the VRJP is a mixture of strongly recurrent Markov chains.

This works as well for any graph with bounded degree.

The proof is based on the following estimate which is valid for any finite graph G and any conductances $(W_{i,j})$.

Lemma

Let $i_0 \in V$ and $x \in V$ and $I(W) = \sqrt{W} \int e^{-W(\cosh(u)-1)} du$.

$$\nu^{i_0,W}\left(e^{t_x/2}\right) \leq \sum_{\sigma: i_0 \to x} \prod_{\{i,j\} \in \sigma} I(W_{i,j}) \prod_{i \in \sigma} e^{\sum_{j \sim i, j \notin \sigma} W_{i,j}}.$$

Sketch of the proof of localization

In coordinates t, the density of $\nu^{i_0,W}$ can be rewritten

$$e^{-\sum_{\{i,j\}\in E} W_{i,j}(\cosh(t_j-t_i)-1)}\sqrt{\sum_{T\in\mathcal{T}_{i_0}}\prod_{(i,j)\in T} W_{i,j}e^{t_j-t_i}}$$

where $\mathcal{T}_{i_0}$ is the set of directed spanning trees directed towards i_0.

$$\begin{aligned}&\sqrt{\sum_{T\in\mathcal{T}_{i_0}}\prod_{(i,j)\in T} W_{i,j}e^{t_j-t_i}}\\ \leq\quad & e^{t_x/2}\sum_{\sigma:i_0\to x}\left(\prod_{\{i,j\}\in\sigma}\sqrt{W_{i,j}}\right)\sqrt{\sum_{T'}\prod_{(i,j)\in T'} W_{i,j}e^{t_j-t_i}}\end{aligned}$$

Integrating the fields on $(t_i)_{i\notin\sigma}$ gives

$$\begin{aligned} &\nu^{i_0,W}(e^{t_x/2}) \\ \le \;& \int \sum_{\sigma} \prod_{\{i,j\}\in\sigma} \sqrt{W_{i,j}} e^{-W_{i,j}(\cosh(t_j-t_i))} \prod_{i\in\sigma} e^{\sum_{j\sim i, j\notin\sigma} W_{i,j}} dt \end{aligned}$$

For the ERRW we need to integrate this inequality on the conductances. Using that for $s<1$, $\nu^{i_0,W}(e^{st_x/2}) \le \nu^{i_0,W}(e^{t_x/2})^s$, it gives exponential decrease of $\mathbb{E}(\nu^{i_0,W}(e^{st_x/2}))$ after integration with respect to W.
With a little more effort we get the exponential decrease of $\mathbb{E}(\nu^{i_0,W}(e^{t_x/2}))$

Corollary

On any graph of bounded degree, for $a_e = a$, there exists a_c such that for $a < a_c$, the ERRW with initial weights a is a mixture of strongly recurrent random walks.

On $\mathbb{Z}^d$, $d \geq 3$ for constant $W_{i,j} = \beta$, Disertori, Spencer, Zirnbauer proved delocalization of the field at weak reinforcement. They proved that for any $m > 0$ there exists β_c such that if $\beta > \beta_c$, then

$$\nu^{0,\beta}(\cosh(t_x)^m)$$

is bounded.

Corollary

$d \geq 3$, and $W_{i,j} = \beta$. There exists $\beta_c(d)$ such that for $\beta > \beta_c(d)$ the VRJP is transient.