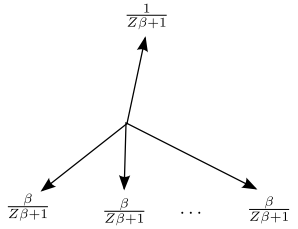
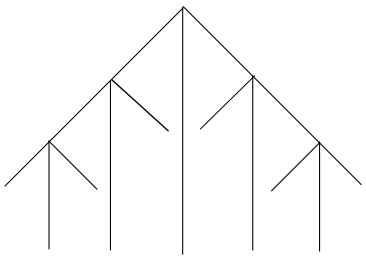
- Monday, April 30, 2012
- Speaker: Alexander Fribergh
- Title: On the monotonicity of the speed of biased random walk on a Galton-Watson tree without leaves
- Note taker: Xiaoqin Guo

1. Model
   $Z$ : offspring distribution of the G-W tree. Assume $EZ > 1$ (super critical)
   $p_k := P(Z = k)$. (If $p_0 > 0$, the tree has "leaves".)
   Biased random walk $(X_n^\beta)$, $\beta > 0$:




2. Q: transient/recurrent?
   -A: (Lyons 91) $\lim_{n\to\infty} |X_n^\beta| = \infty$ if $\beta > 1/E[Z]$.
   Q:Speed?
   -A: (Lyons, Pemantle, Peres 96) $\lim \frac{|X_n^\beta|}{n} = v(\beta)$ almost surely.
   In this talk, we will focus on how $v(\beta)$ depends on $\beta$.
   **Open problem**(Lyons-Pemantle-Peres 96):
   If $p_0 = 0$, is $v(\beta)$ increasing ?

   *Remark:* Example (Deterministic tree without leaves. $v$ is not increasing) : For a binary tree+spikes:



   - $\beta = \frac{1}{2} + \epsilon$, then $v \approx 0$;
   - $\beta = 1 - \epsilon$, $v \approx 0$;
   - $\beta = \frac{3}{4}$, $v > 0$.

3. **Theorem**(BenArous, F., Sidoravicius)
Assume $p_0 = 0$. If $\beta > 717$, then $v(\beta)$ is increasing.
*Remarks:*

- (BenArous, Hu, Olla, Zeiotuni)When $p_0 = 0$, $\frac{\mathrm{d}v}{\mathrm{d}\beta}(\frac{1}{EZ}) = D_0 > 0$.
- (Aidekon)Explicit expression for $v(\beta)$.($\Rightarrow$ $v \nearrow$ for $\beta \geq 2$. use environment viewed from the particle.)

4. The proof of the Theorem.

- *Idea does not work*: take two walks $X^\beta, X^{\beta+\epsilon}$ and couple them to stay as long as possible.
- *Idea that works*: take a third walk $Y$ which is a $\beta$-biased random walk on $\mathbb{Z}$ such that (i)When $X^\beta$ and $X^{\beta+\epsilon}$ decouple, $X^{\beta+\epsilon}$ goes down and $X^\beta$ goes up; (ii)If $Y$ goes from $x$ to $x+1$, then both $X^{\beta+\epsilon}$ and $X^\beta$ go down.

*Why introduce $Y$?*
–They have common regeneration times!
$\tau_1^{\mathbb{Z}}$ :=the regeneration time of $Y$. (When $\beta$ is large, $\tau_1^{\mathbb{Z}}$ is small.)

$$v(\beta) = \frac{EX^\beta_{\tau_1^{\mathbb{Z}}}}{E\tau_1^{\mathbb{Z}}}, \qquad v(\beta+\epsilon) = \frac{EX^{\beta+\epsilon}_{\tau_1^{\mathbb{Z}}}}{E\tau_1^{\mathbb{Z}}}.$$

We only need to compare $EX^\beta_{\tau_1^{\mathbb{Z}}}$ and $EX^{\beta+\epsilon}_{\tau_1^{\mathbb{Z}}}$.

*Proof:*
Let $\Delta_n = |X^{\beta+\epsilon}_{n+1}| - |X^\beta_{n+1}| - (|X^{\beta+\epsilon}_n| - |X^\beta_n|) \in \{-2, 0, 2\}$. If $Y$ goes forward at time $n$, $\Delta_n = 0$. When $X^\beta$ and $X^{\beta+\epsilon}$ decouple, $\Delta_n = 2$.
Case 1 ("$C$"): $X^\beta$ and $X^{\beta+\epsilon}$ stay coupled till $\tau_1^{\mathbb{Z}}$, the difference of the walks $\Delta = 0$.
Case 2 ("$D_1$"): they decouple and $Y$ goes back only once before $\tau_1$, then $\Delta = 2$.
.....
Case $k$("$D_k$"): $\Delta \geq 2 - 2(k-1)$.

$$E[|X^{\beta+\epsilon}_{\tau_1^{\mathbb{Z}}}| - |X^\beta_{\tau_1^{\mathbb{Z}}}|] \geq 0P(C) + 2P(D_1) + \sum_{k\geq 2}[2-2(k-1)]P(D_k).$$

Note that

$$\begin{aligned} P(D_k) &= P(\text{decouple, } Y \text{ goes back } k \text{ times before } \tau_1^{\mathbb{Z}}) \\ &\approx P(\textit{decouple})[P(Y \text{ back})]^{k-1}, \end{aligned}$$

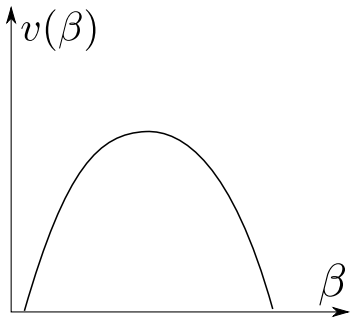
so

$$E[|X^{\beta+\epsilon}_{\tau_1^{\mathbb{Z}}}| - |X^{\beta}_{\tau_1^{\mathbb{Z}}}|] \geq 2P(decouple) + P(decouple)\sum_{k\geq 2}[2-2(k-1)]\beta^{-(k-1)}$$

$$\approx P(decouple)[1-\Theta(\beta^{-1})] \overset{\beta \text{ large}}{>} 0. \quad \square$$

5. Open problems:

   - Conjecture: monotonicity for $\beta \geq 1$, $\beta \geq \frac{1}{EZ}$?
   - This is a "baby problem" of: for a GW tree with $p_0 > 0$, show:




   - Same question for biased random walks on percolation cluster.
   - Is there a tree/graph without leaves where $v(\beta)$ is not increasing for $\beta > 1$?