Spanning Trees of Graphs on Surfaces and the Intensity of Loop-erased Random Walk

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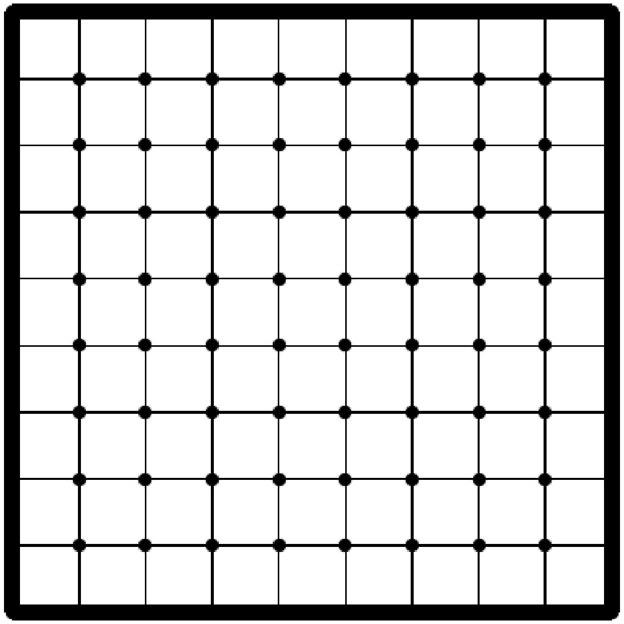
(Brown)

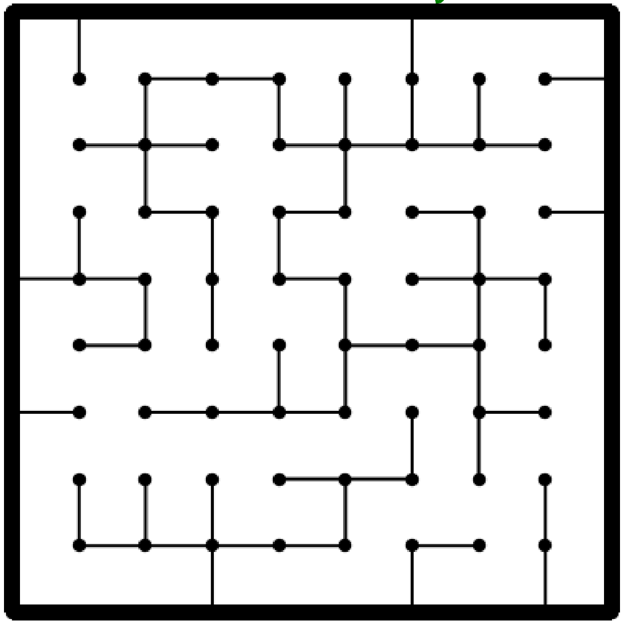
David Wilson
(Microsoft)

arxiv: 1107.3377

Square Grid with Wired Boundary



Uniform Spanning Tree



Uniform Spanning Tree

Path in tree is loop-erased random walk (LERW)

$$WSF(\mathbb{Z}^d) = \lim_{\text{wired box}\to} UST$$

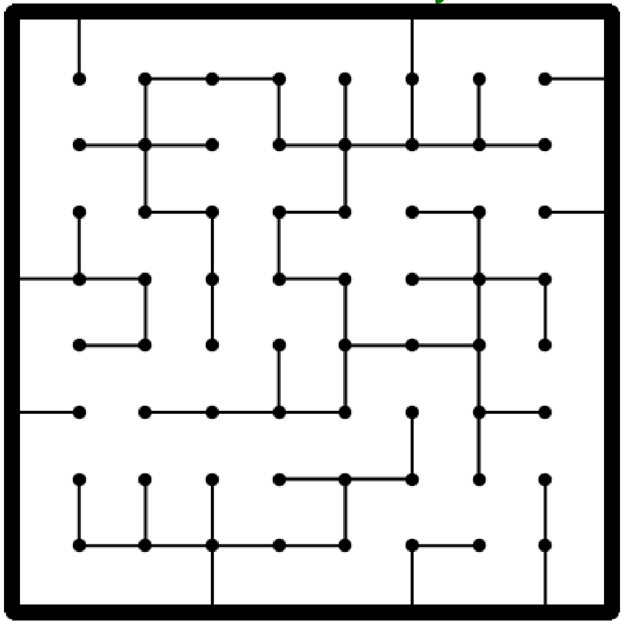
$$FSF(\mathbb{Z}^d) = \lim_{\text{free box}\to} UST$$

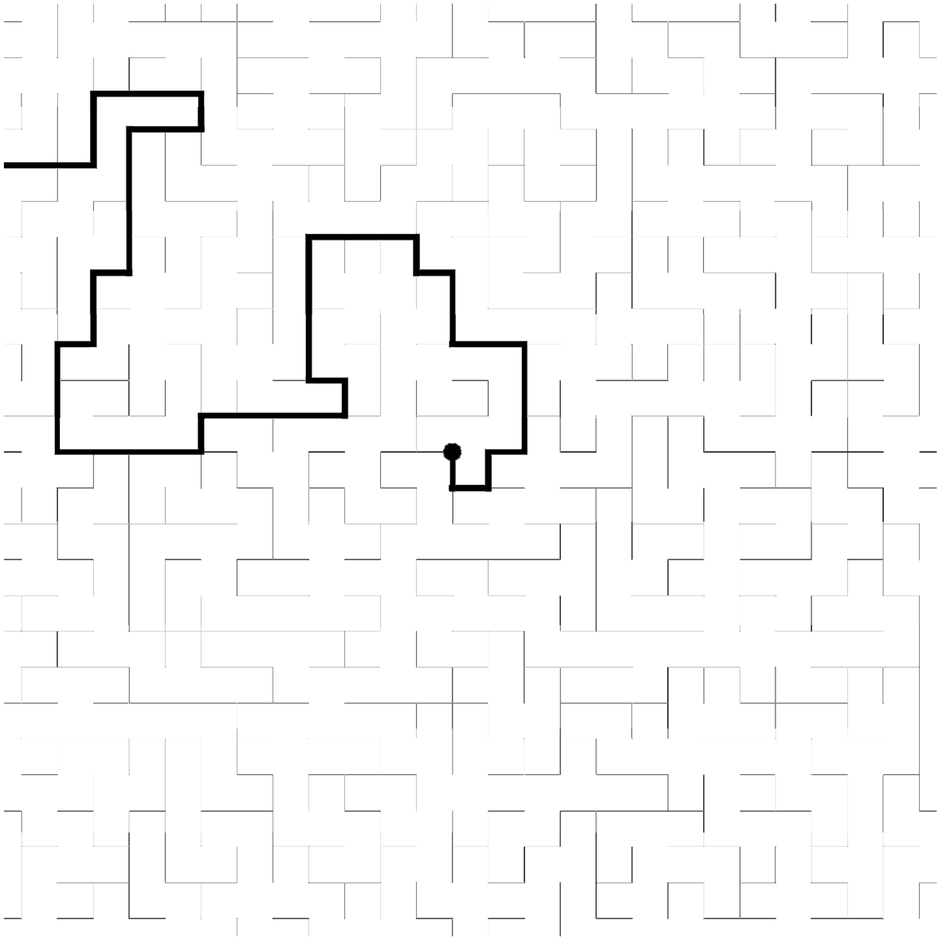
$$WSF(\mathbb{Z}^d) = FSF(\mathbb{Z}^d)$$

Tree when $d \leq 4$

Forest when $d > 4$

(Pemantle)

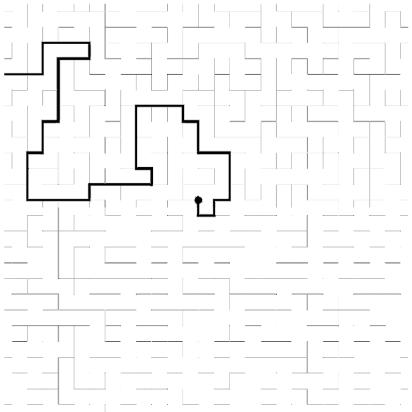


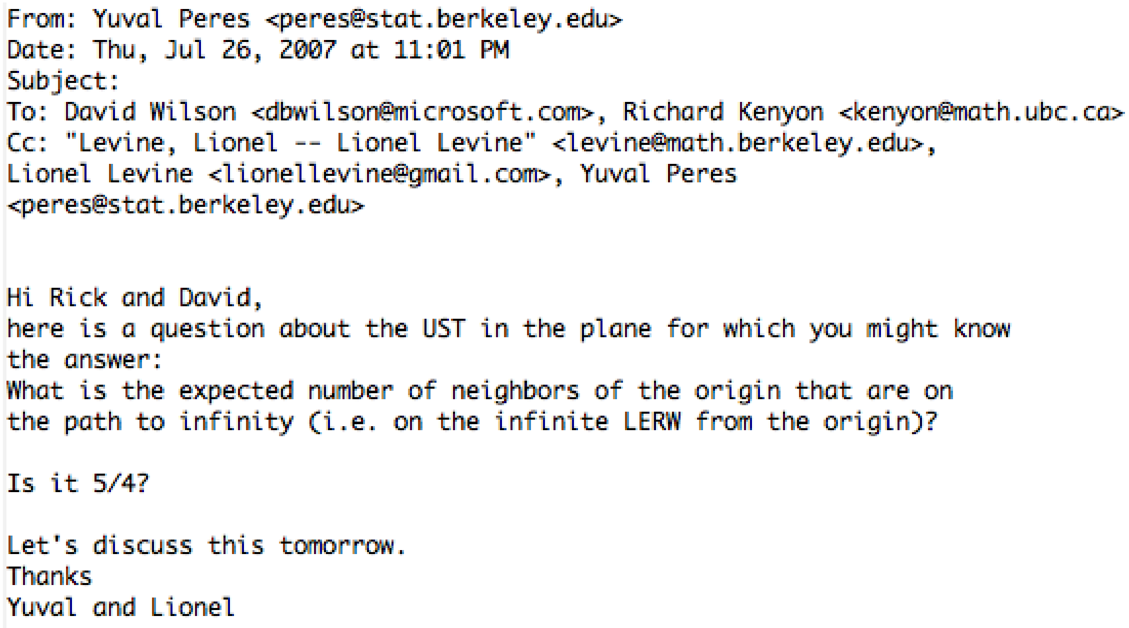


Only one path to ∞

(Benjamini-Lyons-Peres-Schramm)

LERW($\mathbb{Z}^2$)





From: Yuval Peres <peres@stat.berkeley.edu>
Date: Thu, Jul 26, 2007 at 11:01 PM
Subject:
To: David Wilson <dbwilson@microsoft.com>, Richard Kenyon <kenyon@math.ubc.ca>
Cc: "Levine, Lionel -- Lionel Levine" <levine@math.berkeley.edu>,
Lionel Levine <lionellevine@gmail.com>, Yuval Peres
<peres@stat.berkeley.edu>

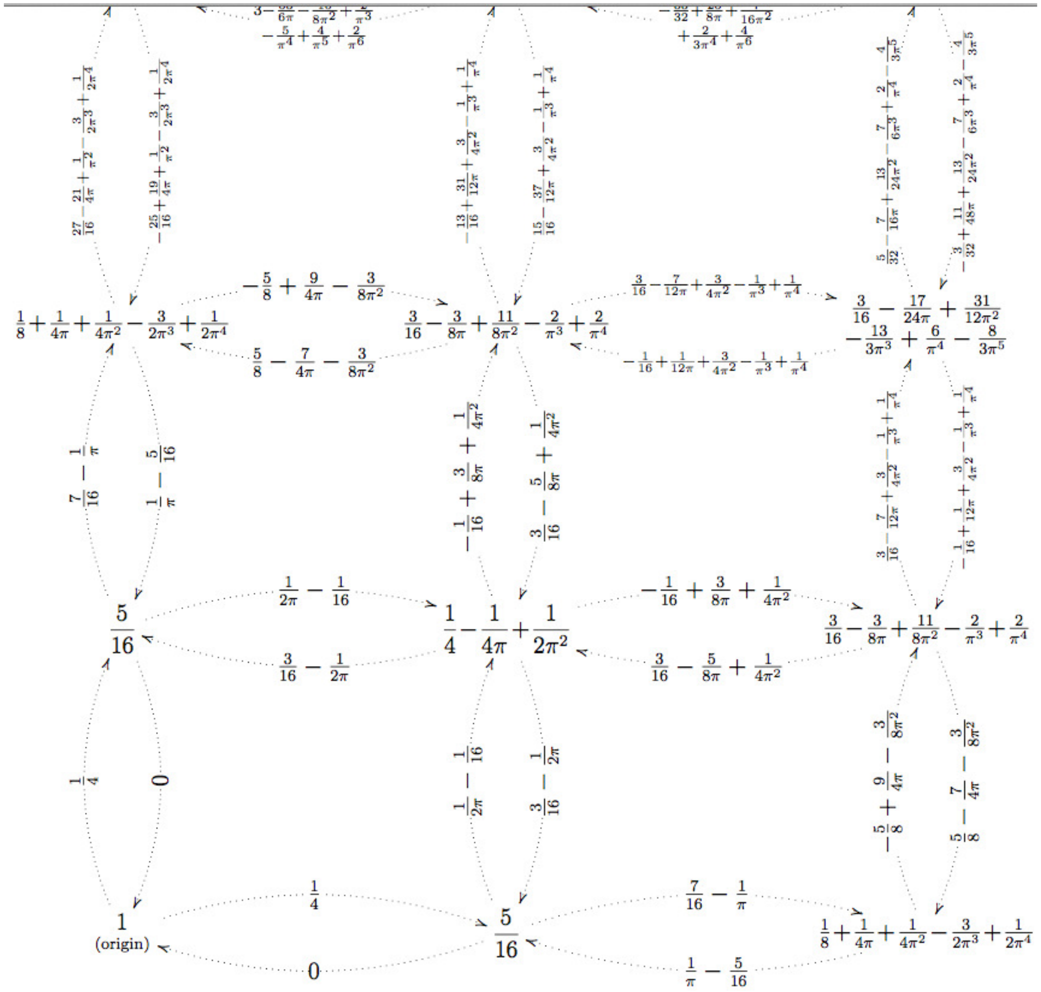
Hi Rick and David,
here is a question about the UST in the plane for which you might know the answer:
What is the expected number of neighbors of the origin that are on the path to infinity (i.e. on the infinite LERW from the origin)?

Is it 5/4?

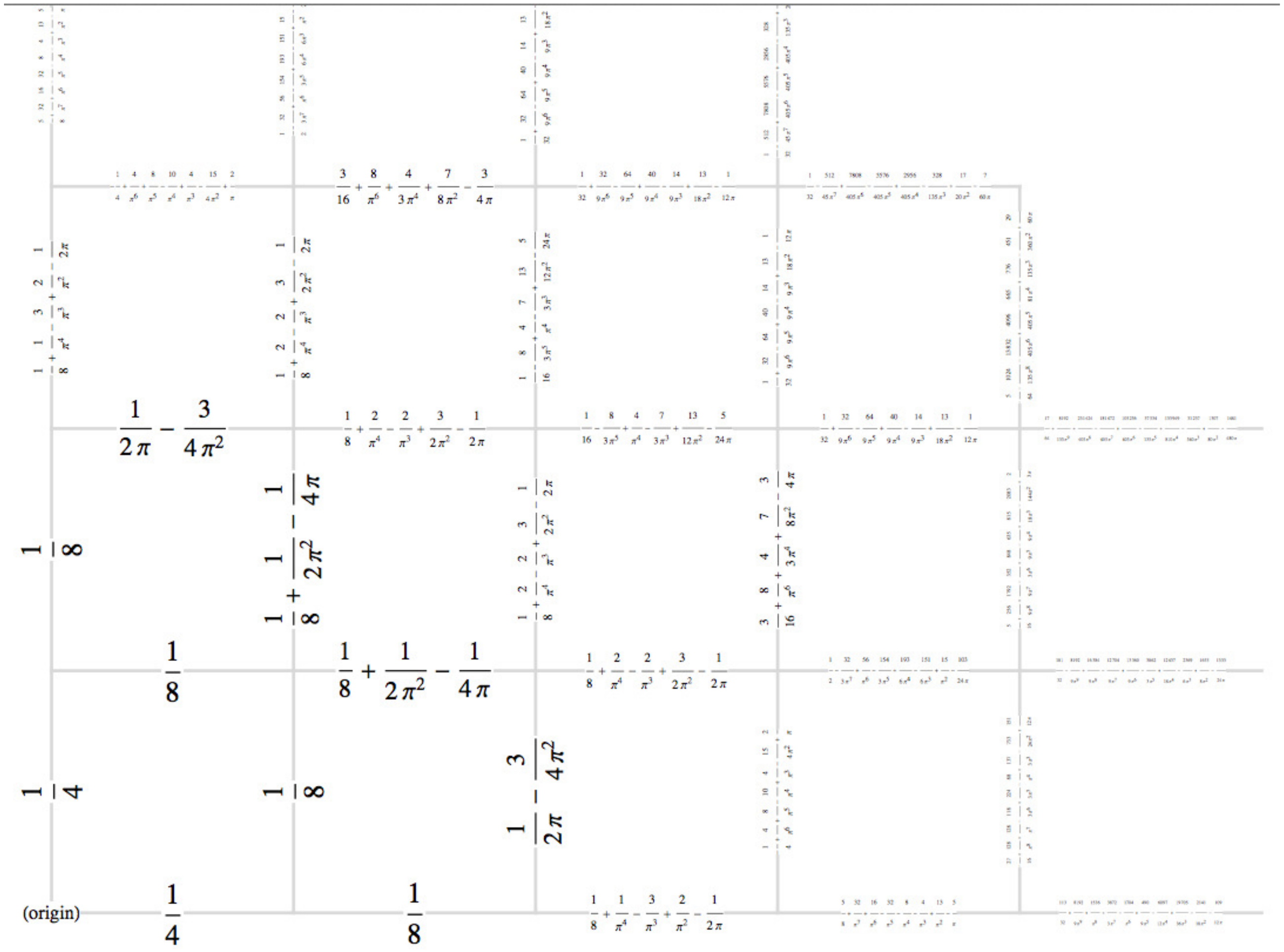
Let's discuss this tomorrow.
Thanks
Yuval and Lionel

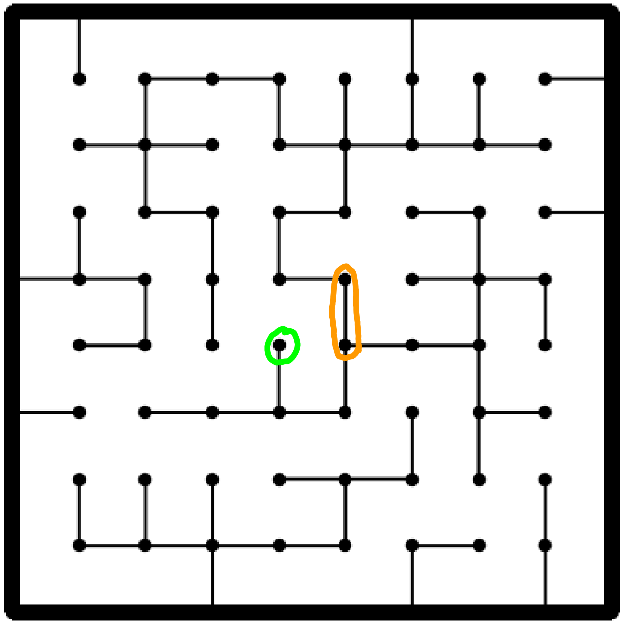
(Also predicted by Poghosyan and Priezzhev)

(Based on previous calculations on "abelian sandpile model" by Majumdar-Dhar, Priezzhev, Grassberger.)

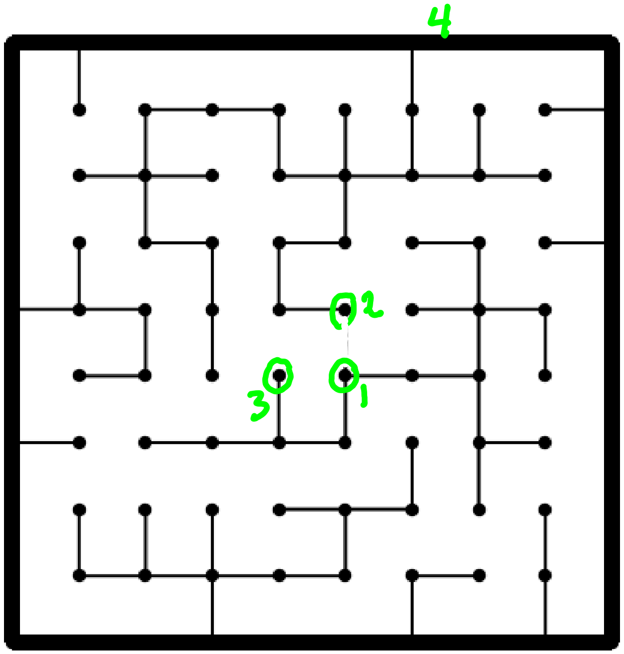


(5/16 independently proved
by Poghosyan, Priezzhev, Ruelle)


(origin)
$\frac{1}{4}$
$\frac{1}{8}$
$\frac{1}{8}+\frac{1}{\pi^4}-\frac{3}{\pi^3}+\frac{2}{\pi^2}-\frac{1}{2\pi}$
$\frac{1}{8}+\frac{1}{2\pi^2}-\frac{1}{4\pi}$
$\frac{1}{8}+\frac{2}{\pi^4}-\frac{2}{\pi^3}+\frac{3}{2\pi^2}-\frac{1}{2\pi}$
$\frac{1}{2\pi}-\frac{3}{4\pi^2}$
$\frac{1}{16}+\frac{8}{3\pi^5}-\frac{4}{\pi^4}+\frac{7}{3\pi^3}+\frac{13}{12\pi^2}-\frac{5}{24\pi}$
$\frac{3}{16}+\frac{8}{\pi^6}+\frac{4}{3\pi^4}+\frac{7}{8\pi^2}-\frac{3}{4\pi}$

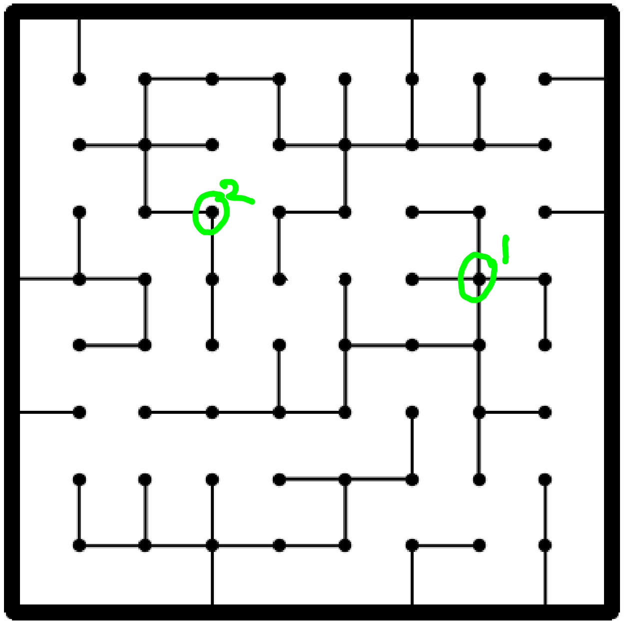


Spanning tree in which path from start to boundary uses given directed edge



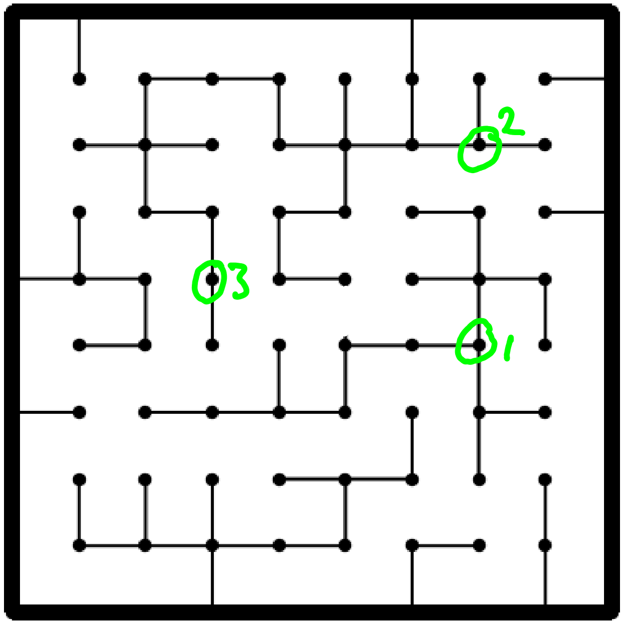
grove of type $1,3|2,4$

$$\Pr[\text{LERW uses directed edge}] = \frac{Z(1,3|2,4)}{Z}$$



grove of type 1/2

Kirchhoff: $R_{1,2} = \frac{Z(1|2)}{Z}$



grove of type 1|2,3

$$\frac{Z(1|23)}{Z} = \frac{1}{2}R_{1,2} + \frac{1}{2}R_{1,3} - \frac{1}{2}R_{2,3}$$

$$\frac{Z(1|2|3)}{Z} = \frac{R_{1,2}R_{1,3}+R_{1,2}R_{2,3}+R_{1,3}R_{2,3}}{2} - \frac{R_{1,2}^2+R_{1,3}^2+R_{2,3}^2}{4}$$

4 nodes (or more)

No formula of the above type that holds for general graphs.

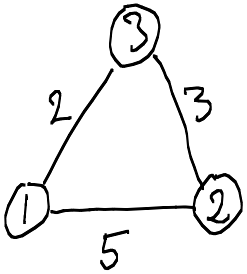
Pairwise resistances do not determine $\frac{Z(1,2|3,4)}{Z}$ for general graphs.

If graph is planar and all nodes on the same face,

then $\frac{Z(\sigma)}{Z}$ is a polynomial in the $R_{i,j}$'s.

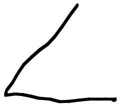
(KW)

Matrix-Tree Theorem (Kirchhoff)



$$\Delta = \begin{array}{cccc} & 1 & 2 & 3 \\ 1 & 7 & -5 & -2 \\ 2 & -5 & 8 & -3 \\ 3 & -2 & -3 & 5 \end{array}$$

Laplacian



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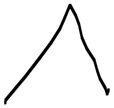

+ 6

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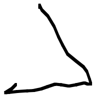


15

= 31

$\det \Delta = 0$

$\det \Delta^{1,2}_{1,2} = 56-25 = 31$

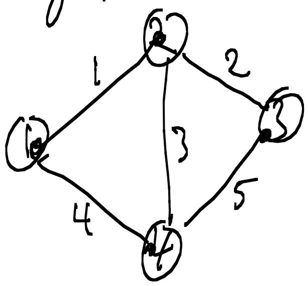
$\det \Delta^{1,3}_{1,3} = 35-4 = 31$

$\det \Delta^{2,3}_{2,3} = 40-9 = 31$

Matrix Tree Theorem

$$\Delta = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & +5 & -1 & 0 & -4 \\ 2 & -1 & +6 & -2 & -3 \\ 3 & 0 & -2 & +7 & -5 \\ 4 & -4 & -3 & -5 & +12 \end{array}$$

graph

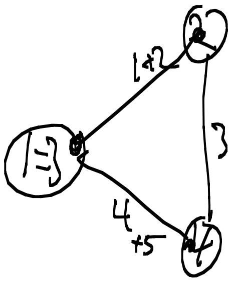


$\det \Delta = 0$

Kirchhoff: $\det \Delta^{1,2,3}_{1,2,3} = 210 - 20 - 7 = 183$

$=$ weighted sum of spanning trees

graph with nodes 1 and 3
merged



$$\tilde{\Delta} = \begin{array}{c|ccc} & 1{=}3 & 2 & 4 \\ \hline 3{=}1 & +12 & -3 & -9 \\ 2 & -3 & +6 & -3 \\ 4 & -9 & -3 & +2 \end{array}$$

$$\det \tilde{\Delta} = 0$$

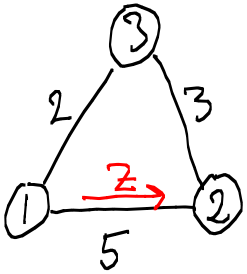
Kirchhoff: $\det \tilde{\Delta}^{1=3,2}_{1=3,2} = 63$

= weighted sum of spanning trees of glued graph

= groves of type 1|3 of original graph

$R_{1,3} = 63/183 = 21/61$

Line Bundle Laplacian



$$\Delta = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 7 & -5/z & -2 \\ 2 & -5z & 8 & -3 \\ 3 & -2 & -3 & 5 \end{array}$$

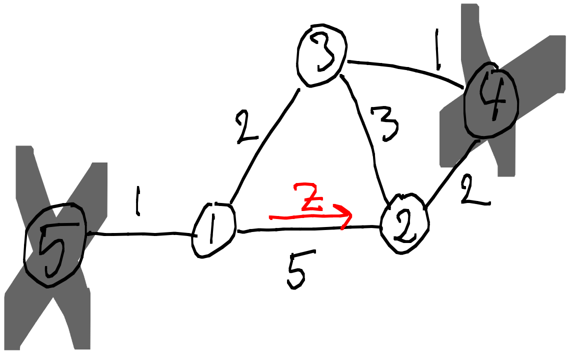
Forman:

product of edge weights

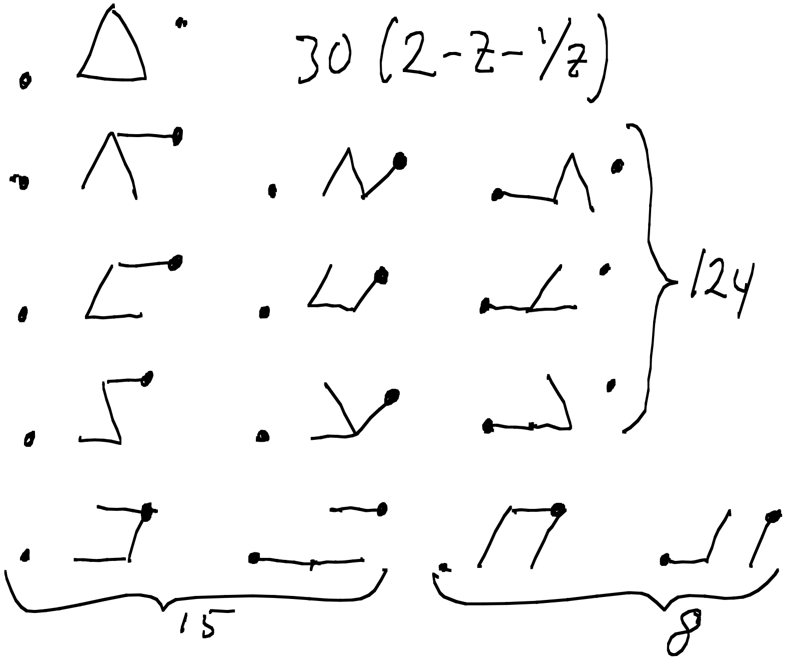
monodromy of cycle

$$\det\Delta = 30\left(2 - z - 1/z\right)$$

= weighted sum of cycle-rooted spanning forests

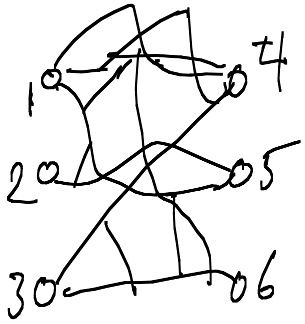


$$\Delta = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 8 & -5/z & -2 & 0 & -1 \\ 2 & -5z & 10 & -3 & -2 & 0 \\ 3 & -2 & -3 & 6 & -1 & 0 \\ 4 & 0 & -2 & -1 & 3 & 0 \\ 5 & -1 & & & & 1 \end{array}$$



$$\det\Delta = 480 - 30z - 30/z$$
$$-72 - 150 - 40$$
$$= 30(2-z-1/z) + 158$$

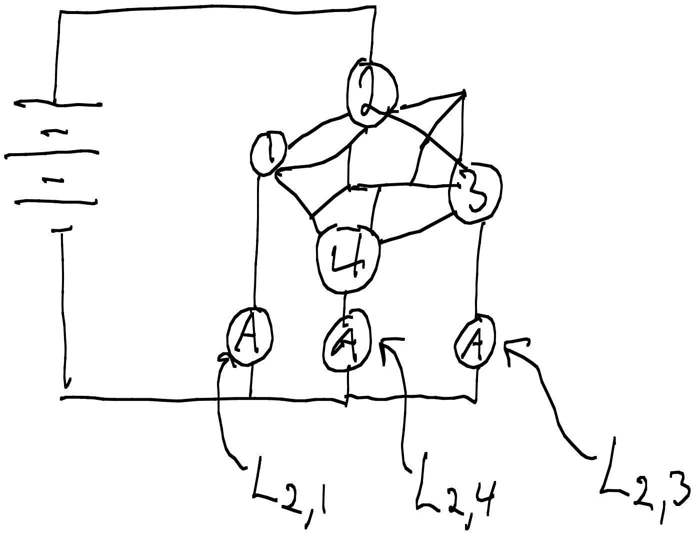
Cartis - Ingerman - Morrow



general graph with
2n vertices

$$\det L^{4,5,6}_{1,2,3} = \frac{\begin{aligned} &Z(14|25|36) - Z(14|26|35) \\ &+ Z(15|26|34) - Z(15|24|36) \\ &+ Z(16|24|35) - Z(16|25|34) \end{aligned}}{Z(1|2|3|4|5|6)}$$

Response matrix $L_{i,j}$: apply voltage at i; measure current at j.



Fact: $L_{i,j}=L_{j,i}$

$\sum_j L_{i,j} = 0$

$\binom{n}{2}$ $L_{i,j}$ variables

$\binom{n}{2}$ $R_{i,j}$ variables

Computing Response Matrix

$$\Delta = \begin{array}{c@{}c} & \begin{matrix} \text{nodes} & \text{internal} \end{matrix} \\ \begin{matrix} \text{nodes} \\ \text{internal} \end{matrix} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array} \begin{bmatrix} v_E \\ v_I \end{bmatrix} = \begin{bmatrix} i_E \\ 0 \end{bmatrix}$$

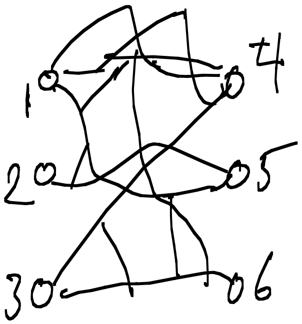
$$v_I = -D^{-1} C v_E$$

$$i_E = A v_E - B D^{-1} C v_E$$

$$\Lambda = A - B D^{-1} C = \Lambda^*$$

$$L = -\Lambda$$

$\mathcal{L}$ (if a function of z)



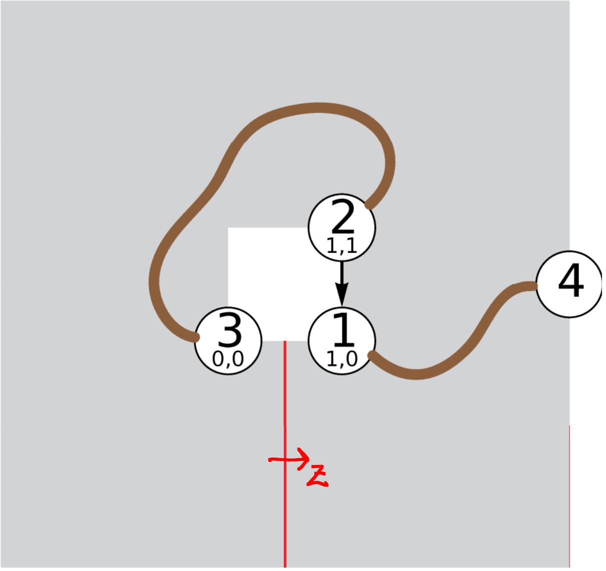
general graph with
2n vertices

$$\det \mathcal{L}^{4,5,6}_{1,2,3} = \frac{\mathcal{Z}\left(\begin{array}{c|c|c}4&5&6\\1&2&3\end{array}\right) - \mathcal{Z}\left(\begin{array}{c|c|c}4&6&5\\1&2&3\end{array}\right) + \mathcal{Z}\left(\begin{array}{c|c|c}5&6&4\\1&2&3\end{array}\right) - \mathcal{Z}\left(\begin{array}{c|c|c}5&4&6\\1&2&3\end{array}\right) + \mathcal{Z}\left(\begin{array}{c|c|c}6&4&5\\1&2&3\end{array}\right) - \mathcal{Z}\left(\begin{array}{c|c|c}6&5&4\\1&2&3\end{array}\right)}{\mathcal{Z}(1|2|3|4|5|6)}$$

includes parallel transports

cycle rooted groves

(proof similar to C-I-M proof, but starts with Forman's MTT)



$$\det \mathcal{L}^{2,4}_{1,3} = \dddot{\tilde{Z}}\left(\begin{array}{c|c}2&4\\1&3\end{array}\right) - \dddot{\tilde{Z}}\left(\begin{array}{c|c}2&4\\3&1\end{array}\right)$$

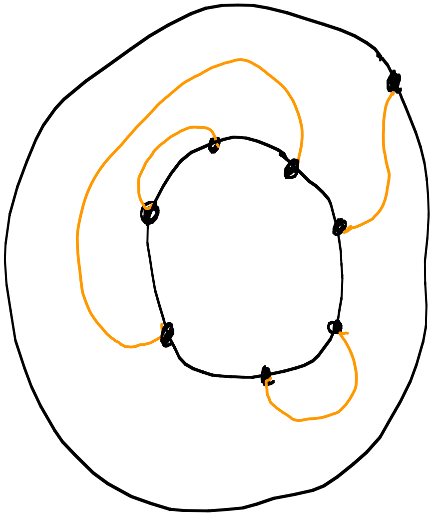
$$\det \mathcal{L}^{1,4}_{3,2} = \dddot{\tilde{Z}}\left(\begin{array}{c|c}1&4\\3&2\end{array}\right) - \dddot{\tilde{Z}}\left(\begin{array}{c|c}1&4\\2&3\end{array}\right)$$

$$\det \mathcal{L}^{3,4}_{2,1} = \dddot{\tilde{Z}}\left(\begin{array}{c|c}3&4\\2&1\end{array}\right) - \underbrace{\dddot{\tilde{Z}}\left(\begin{array}{c|c}3&4\\1&2\end{array}\right)}_{= z^2 \dddot{\tilde{Z}}\left(\begin{array}{c|c}1&4\\3&2\end{array}\right)}$$

$$\dddot{\tilde{Z}}\left(\begin{array}{c|c}3&4\\2&1\end{array}\right) \simeq \frac{z^2 \det \mathcal{L}^{2,4}_{1,3} + z^2 \det \mathcal{L}^{1,4}_{3,2} + \det \mathcal{L}^{3,4}_{2,1}}{1-z^2}$$

$$\Downarrow$$

$$\frac{Z(2,3|1,4)}{Z(1|2|3|4)} = L_{1,4}L_{2,3} - L_{1,3}L_{2,4} - L'_{1,2}L_{3,4} - L'_{2,3}L_{1,4} - L'_{3,1}L_{2,4}$$



n nodes, n-1 on inner boundary

$$(n-1) \times C_{\frac{n}{2}-1} = (n-1) \frac{(n-2)!}{(\frac{n}{2})!\,(\frac{n}{2}-1)!}$$

$$= \frac{1}{2} \frac{n}{\frac{n}{2}} (n-1) \frac{(n-2)!}{(\frac{n}{2})!\,(\frac{n}{2}-1)!}$$

$$= \frac{1}{2} \binom{n}{n/2}$$

\# of determinants = # of annular perfect matchings

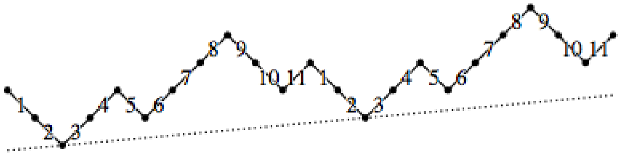
$$\begin{array}{c|ccc}
 & \dddot{\mathscr{Z}}\left[\begin{array}{c|c}2&4\\1&3\end{array}\right] & \dddot{\mathscr{Z}}\left[\begin{array}{c|c}1&4\\3&2\end{array}\right] & \dddot{\mathscr{Z}}\left[\begin{array}{c|c}3&4\\2&1\end{array}\right] \\
\hline
\det \mathscr{L}^{2,4}_{1,3} & 1 & 0 & -1 \\
\det \mathscr{L}^{1,4}_{3,2} & -1 & 1 & 0 \\
\det \mathscr{L}^{3,4}_{2,1} & 0 & -\zeta & 1
\end{array}$$

$$\begin{array}{c|ccc}
 & \det \mathscr{L}^{2,4}_{1,3} & \det \mathscr{L}^{1,4}_{3,2} & \det \mathscr{L}^{3,4}_{2,1} \\
\hline
\dddot{\mathscr{Z}}\left[\begin{array}{c|c}2&4\\1&3\end{array}\right] & 1 & \zeta & 1 \\
\dddot{\mathscr{Z}}\left[\begin{array}{c|c}1&4\\3&2\end{array}\right] & 1 & 1 & 1 \\
\dddot{\mathscr{Z}}\left[\begin{array}{c|c}3&4\\2&1\end{array}\right] & \zeta & \zeta & 1
\end{array} \times \frac{1}{(1-\zeta)^1}$$

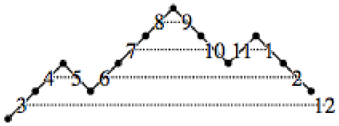
$$\begin{array}{c|cccccccccc}
 & \ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}2&4&6\\1&3&5\end{array}\right] & \ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}4&3&6\\1&2&5\end{array}\right] & \ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}1&3&6\\5&2&4\end{array}\right] & \ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}3&2&6\\5&1&4\end{array}\right] & \ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}5&2&6\\4&1&3\end{array}\right] & \ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}2&1&6\\4&5&3\end{array}\right] & \ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}4&1&6\\3&5&2\end{array}\right] & \ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}1&5&6\\3&4&2\end{array}\right] & \ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}3&5&6\\2&4&1\end{array}\right] & \ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}5&4&6\\2&3&1\end{array}\right] \\
\hline
\det \mathscr{L}^{2,4,6}_{1,3,5} & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\
\det \mathscr{L}^{4,3,6}_{1,2,5} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta & -1 & 0 \\
\det \mathscr{L}^{1,3,6}_{5,2,4} & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
\det \mathscr{L}^{3,2,6}_{5,1,4} & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\det \mathscr{L}^{5,2,6}_{4,1,3} & 0 & 0 & \zeta & -\zeta & 1 & -\zeta & 0 & 0 & -1 & 0 \\
\det \mathscr{L}^{2,1,6}_{4,5,3} & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\det \mathscr{L}^{4,1,6}_{3,5,2} & -1 & 0 & 0 & 0 & 1 & -\zeta & 1 & -1 & 0 & 0 \\
\det \mathscr{L}^{1,5,6}_{3,4,2} & 0 & 0 & 0 & \zeta & -1 & 0 & 0 & 1 & 0 & 0 \\
\det \mathscr{L}^{3,5,6}_{2,4,1} & 0 & 0 & -\zeta & 0 & 0 & 0 & \zeta & -\zeta & 1 & -1 \\
\det \mathscr{L}^{5,4,6}_{2,3,1} & 0 & 0 & 0 & 0 & 0 & \zeta^2 & -\zeta & 0 & 0 & 1
\end{array}$$

$$\begin{array}{c|cccccccccc}
 & \det \mathscr{L}^{2,4,6}_{1,3,5} & \det \mathscr{L}^{4,3,6}_{1,2,5} & \det \mathscr{L}^{1,3,6}_{5,2,4} & \det \mathscr{L}^{3,2,6}_{5,1,4} & \det \mathscr{L}^{5,2,6}_{4,1,3} & \det \mathscr{L}^{2,1,6}_{4,5,3} & \det \mathscr{L}^{4,1,6}_{3,5,2} & \det \mathscr{L}^{1,5,6}_{3,4,2} & \det \mathscr{L}^{3,5,6}_{2,4,1} & \det \mathscr{L}^{5,4,6}_{2,3,1} \\
\hline
\ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}2&4&6\\1&3&5\end{array}\right] & \zeta+1 & \zeta+1 & \zeta^2+\zeta & 2\zeta & \zeta+1 & \zeta^2+\zeta & 2\zeta & 2\zeta & \zeta+1 & 2 \\
\ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}4&3&6\\1&2&5\end{array}\right] & \zeta & 1 & \zeta & \zeta & \zeta & \zeta^2 & \zeta & \zeta & 1 & 1 \\
\ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}1&3&6\\5&2&4\end{array}\right] & \zeta+1 & 2 & \zeta+1 & \zeta+1 & \zeta+1 & 2\zeta & \zeta+1 & \zeta+1 & 2 & 2 \\
\ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}3&2&6\\5&1&4\end{array}\right] & 1 & 1 & \zeta & 1 & 1 & \zeta & \zeta & \zeta & 1 & 1 \\
\ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}5&2&6\\4&1&3\end{array}\right] & 2\zeta & 2\zeta & \zeta^2+\zeta & 2\zeta & \zeta+1 & \zeta^2+\zeta & \zeta^2+\zeta & 2\zeta & \zeta+1 & \zeta+1 \\
\ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}2&1&6\\4&5&3\end{array}\right] & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}4&1&6\\3&5&2\end{array}\right] & \zeta+1 & \zeta+1 & 2\zeta & 2\zeta & \zeta+1 & 2\zeta & \zeta+1 & \zeta+1 & \zeta+1 & 2 \\
\ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}1&5&6\\3&4&2\end{array}\right] & \zeta & \zeta & \zeta & \zeta & 1 & \zeta & \zeta & 1 & 1 & 1 \\
\ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}3&5&6\\2&4&1\end{array}\right] & \zeta^2+\zeta & 2\zeta & \zeta^2+\zeta & \zeta^2+\zeta & 2\zeta & 2\zeta^2 & \zeta^2+\zeta & 2\zeta & \zeta+1 & \zeta+1 \\
\ddddot{\mathscr{Z}}\left[\begin{array}{c|c|c}5&4&6\\2&3&1\end{array}\right] & \zeta & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta^2 & \zeta & \zeta & \zeta & 1
\end{array} \times \frac{1}{(1-\zeta)^2}$$

{1,2,5,9,10,12}
{3,4,6,7,8,11} ⇒



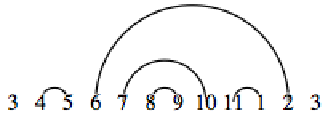
⇒



$\Rightarrow \begin{array}{c|c|c|c|c|c} 5 & 2 & 10 & 9 & 1 & 12 \\ 4 & 6 & 7 & 8 & 11 & 3 \end{array}$

and in the reverse direction,

$1, 11|2, 6|3, 12|4, 5|7, 10|8, 9 \Rightarrow$



$\Rightarrow \begin{array}{c|c|c|c|c|c} 5 & 2 & 10 & 9 & 1 & 12 \\ 4 & 6 & 7 & 8 & 11 & 3 \end{array} \Rightarrow \begin{array}{cccccc} 5, & 2, & 10, & 9, & 1, & 12 \\ 4, & 6, & 7, & 8, & 11, & 3 \end{array}.$

Cyle lemma bijection of Dvoretzky and Motzkin
gives natural association of rows and columns.

Conjugate matrix by a diagonal matrix,
result has all ones on diagonal,
powers of $\xi^{1/n-1}$ in off-diagonal entries
⇒ determinate at $\xi=0$ is 1, matrix invertible.

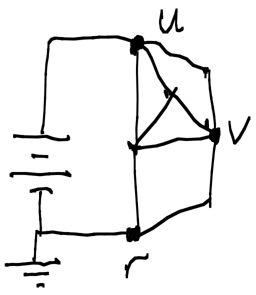
$$\frac{Z(2,3|1,4)}{Z} = \underset{\searrow\, 0}{\frac{Z(2,3|1,4)}{Z(1|2|3|4)}} \times \underset{\searrow\, \infty}{\frac{Z(1|2|3|4)}{Z}}$$

limit as box → $\mathbb{Z}^2$

Green's function

Δ = Laplacian with Dirichlet boundary conditions at r
(row and column r deleted)

$G = \Delta^{-1}$ $\qquad G_{u,v} = G_{v,u}$



$G_{u,v}$ = voltage at v when r is at 0 volts and one unit of current is inserted at u and extracted at r

= average time that random walk from u to r spends at v

Green's function

Δ = line-bundle Laplacian with Dirichlet boundary conditions at n
(row and column n deleted)

$\mathcal{G} = \Delta^{-1}$ $\qquad$ $\mathcal{G}_{u,v} = \mathcal{G}_{v,u}^{*}$

$$\mathcal{G}^{1,2,\dots,n-1}_{1,2,\dots,n-1} = -\left(\mathcal{L}^{1,2,\dots n-1}_{1,2,\dots,n-1}\right)^{-1}$$

$$\frac{\det \mathcal{L}^{S,Q}_{R,Q}}{\det \mathcal{L}^{1,\dots,n-1}_{1,\dots,n-1}} = \det \mathcal{G}^{S,T}_{R,T}$$

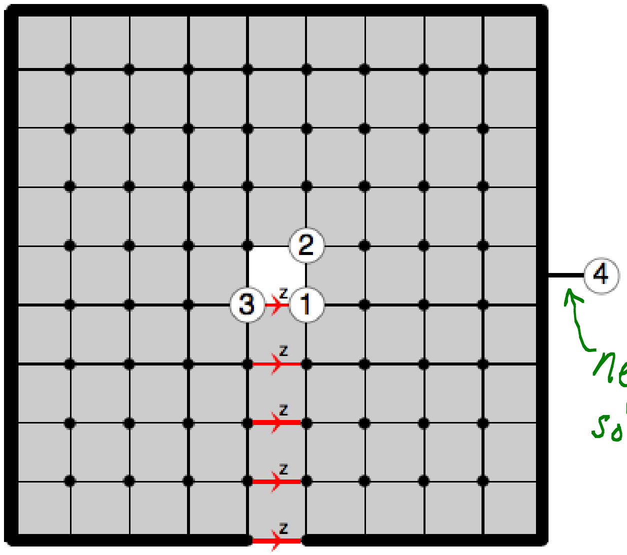
$\mathcal{G}_{i,n} = 1$

Q,R,S,T disjoint, cover $1,2,\dots,n$

$$\frac{Z(2,3|1,4)}{Z} = \underbrace{\overset{\nearrow\infty}{G_{2,3}} - \overset{\nearrow\infty}{G_{1,3}}}_{\text{converges to difference in ``potential kernel''}} - \underbrace{\overset{\nearrow\infty}{G'_{1,2}} - \overset{\nearrow\infty}{G'_{2,3}} - \overset{\nearrow\infty}{G'_{3,1}}}_{\text{converges}} \quad \text{as box} \to \mathbb{Z}^2$$

Potential kernel of $\mathbb{Z}^2$

$20-\frac{184}{3\pi}$	$\frac{40}{\pi}-\frac{49}{4}$	$3-\frac{118}{15\pi}$	$\frac{12}{5\pi}-\frac{1}{4}$	$\frac{176}{105\pi}$
$\frac{17}{4}-\frac{12}{\pi}$	$\frac{23}{3\pi}-2$	$\frac{1}{4}+\frac{2}{3\pi}$	$\frac{23}{15\pi}$	$\frac{12}{5\pi}-\frac{1}{4}$
$1-\frac{2}{\pi}$	$\frac{2}{\pi}-\frac{1}{4}$	$\frac{4}{3\pi}$	$\frac{1}{4}+\frac{2}{3\pi}$	$3-\frac{118}{15\pi}$
$\frac{1}{4}$	$\frac{1}{\pi}$	$\frac{2}{\pi}-\frac{1}{4}$	$\frac{23}{3\pi}-2$	$\frac{40}{\pi}-\frac{49}{4}$
0 (origin)	$\frac{1}{4}$	$1-\frac{2}{\pi}$	$\frac{17}{4}-\frac{12}{\pi}$	$20-\frac{184}{3\pi}$



negative conductance chosen so that $G_{3,4} = 0$

1	(1,0)
2	(1,1)
3	(0,0)
4	"new infinity"

auxillary node and edge does not change $\frac{Z(2,3|1,4)}{Z}$

Now the G's converge

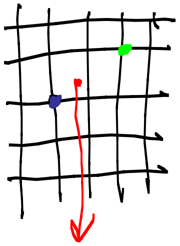
$$S_{k,\ell} = \begin{cases} 1 & \text{there is a zipper edge directed from } k \text{ to } \ell \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Delta(z) = \Delta_0 + (1 - z^{-1})S + (1 - z)S^*$, so

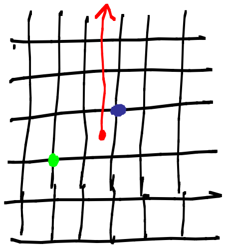
$$\begin{aligned} \Delta(z)^{-1} &= \left(\Delta_0(1 + (1 - z^{-1})\Delta_0^{-1}S + (1 - z)\Delta_0^{-1}S^*)\right)^{-1} \\ &= \Delta_0^{-1} - (1 - z^{-1})\Delta_0^{-1}S\Delta_0^{-1} - (1 - z)\Delta_0^{-1}S^*\Delta_0^{-1} + O((z-1)^2) \\ \mathscr{G}_{u,v} &= G_{u,v} - (z - 1) \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(G_{u,k}G_{\ell,v} - G_{u,\ell}G_{k,v}) + O((z-1)^2) \end{aligned}$$

$$G'_{u,v} = \partial_z \mathscr{G}_{u,v}\big|_{z=1} = - \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(G_{u,k}G_{\ell,v} - G_{u,\ell}G_{k,v})$$

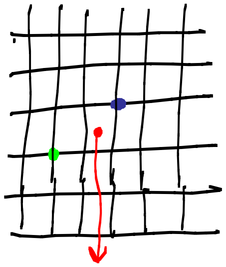
$$\bar{G}'^{\mathbb{L}}_{u,v} = - \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(\bar{G}^{\mathbb{L}}_{u,k}\bar{G}^{\mathbb{L}}_{\ell,v} - \bar{G}^{\mathbb{L}}_{u,\ell}\bar{G}^{\mathbb{L}}_{k,v})$$



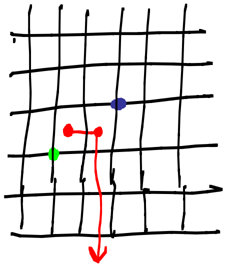
$$G'_{(0,0),(2,1)} = G'^{\uparrow}_{(1,1),(-1,0)}$$



$$= G'_{(1,1),(-1,0)} + G_{(1,1),(-1,0)}$$

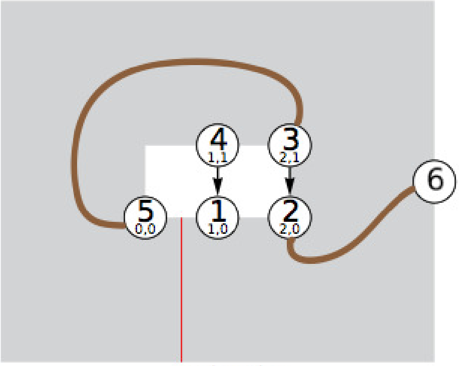


$$= \underbrace{G'_{(2,1),(0,0)}}_{\substack{// \\ -G'_{(0,0),(2,1)}}} + \left(G_{(1,1),(0,0)}\, G_{(0,1),(-1,0)} - G_{(1,1),(0,1)}\, G_{(0,0),(-1,0)}\right) + G_{(1,1),(-1,0)}$$



$$G'_{(0,0),(2,1)} = \frac{-5}{32} + \frac{1}{\pi} + \frac{1}{2\pi^2}$$

Directed edge (2,1)→(2,0)



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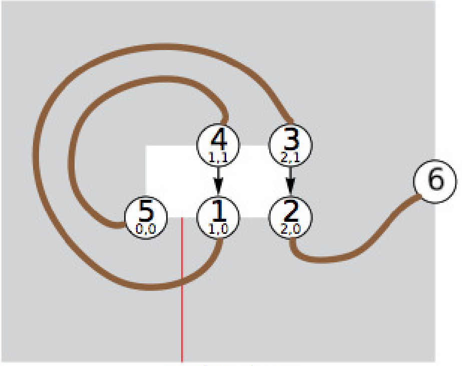
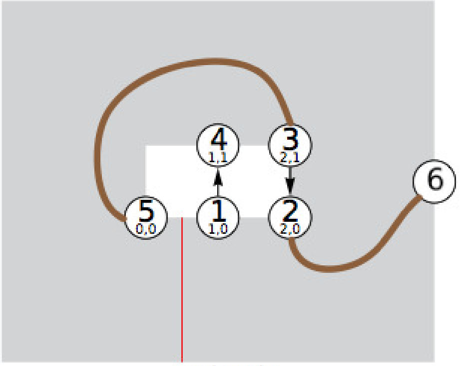


54|13|26



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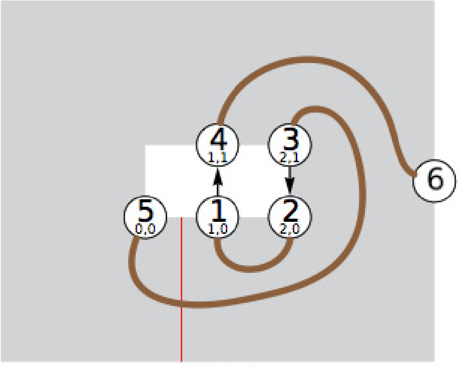

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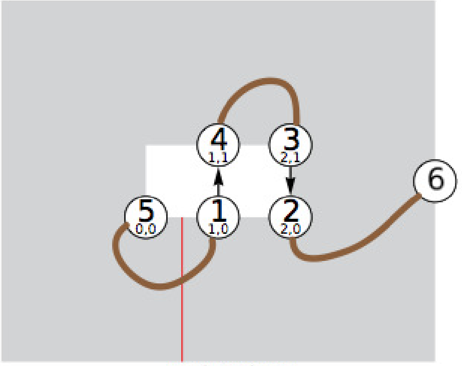

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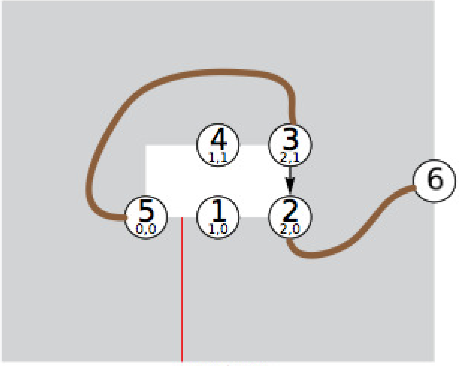

53|26


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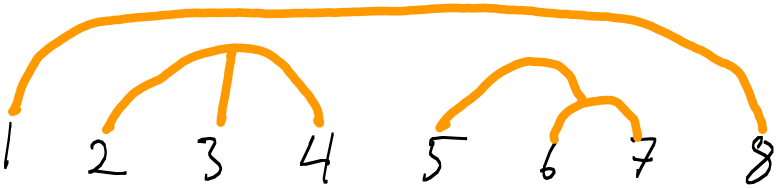
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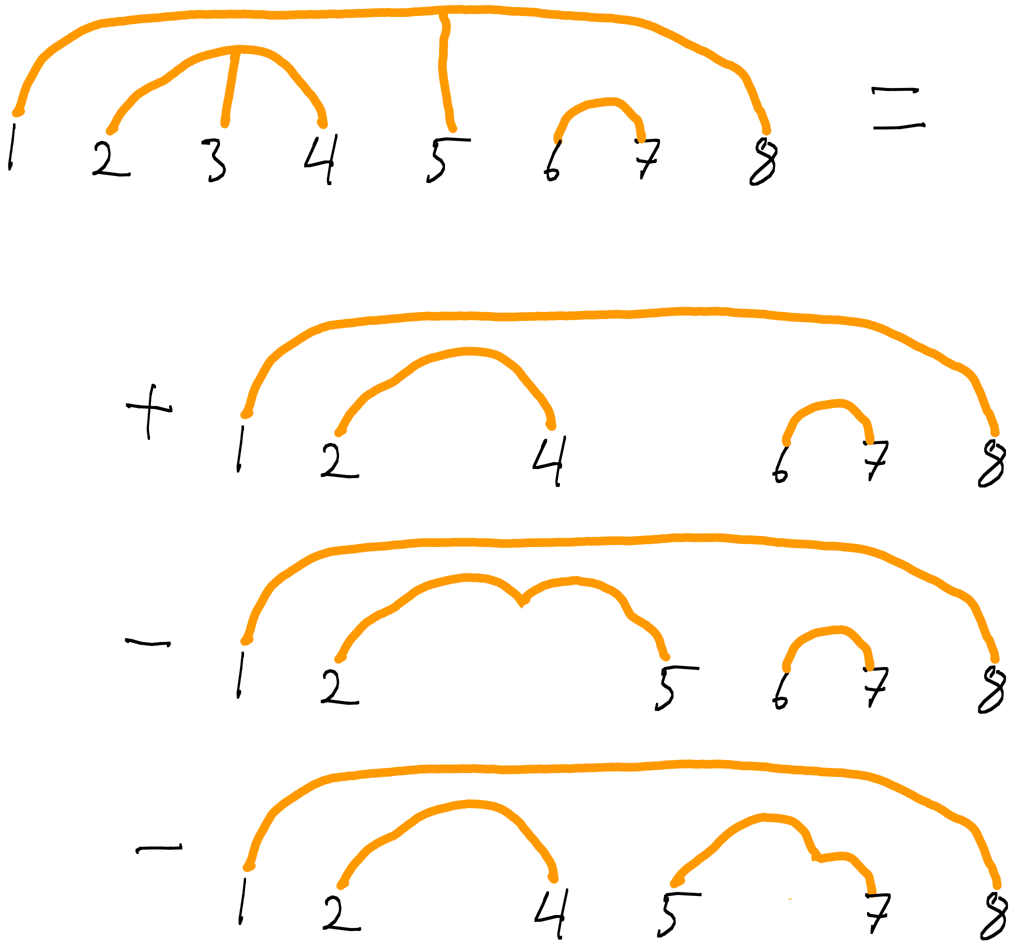
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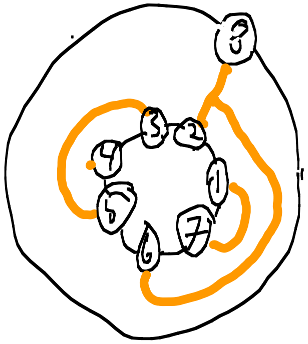

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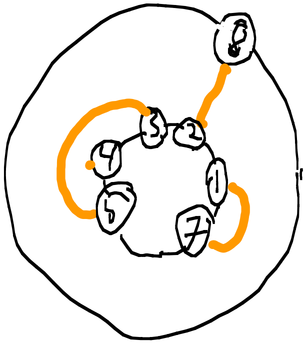

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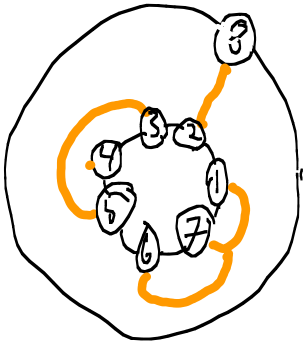

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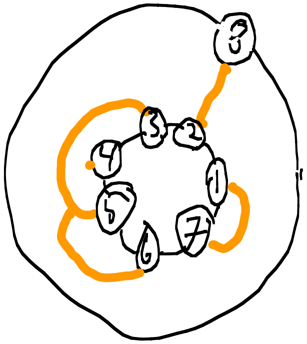

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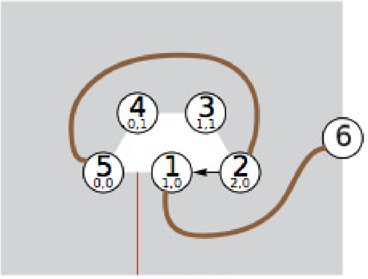
= 1,7/3,5/4/2,8 − 1,6/3,5/4/2,8 − 1,7/3,6/4/2,8

+ 1,7/3,6/4,5/2,8



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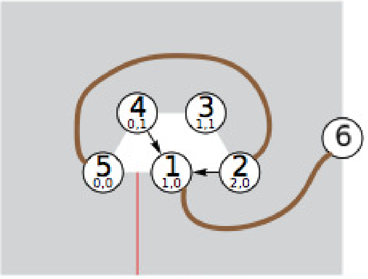

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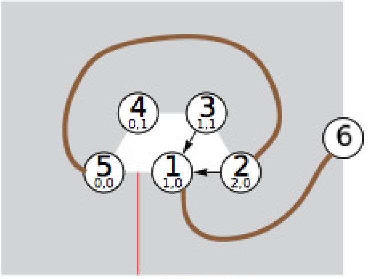

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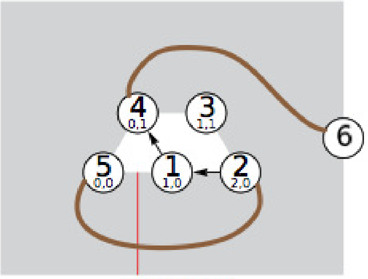

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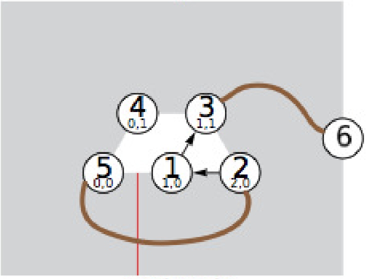

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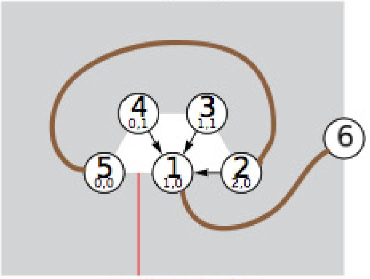

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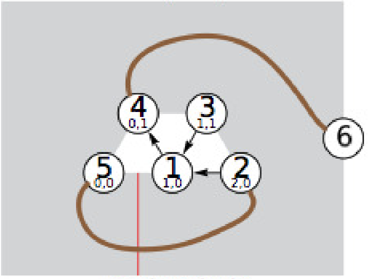

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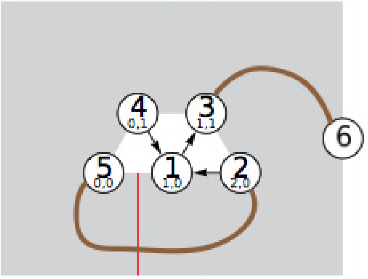

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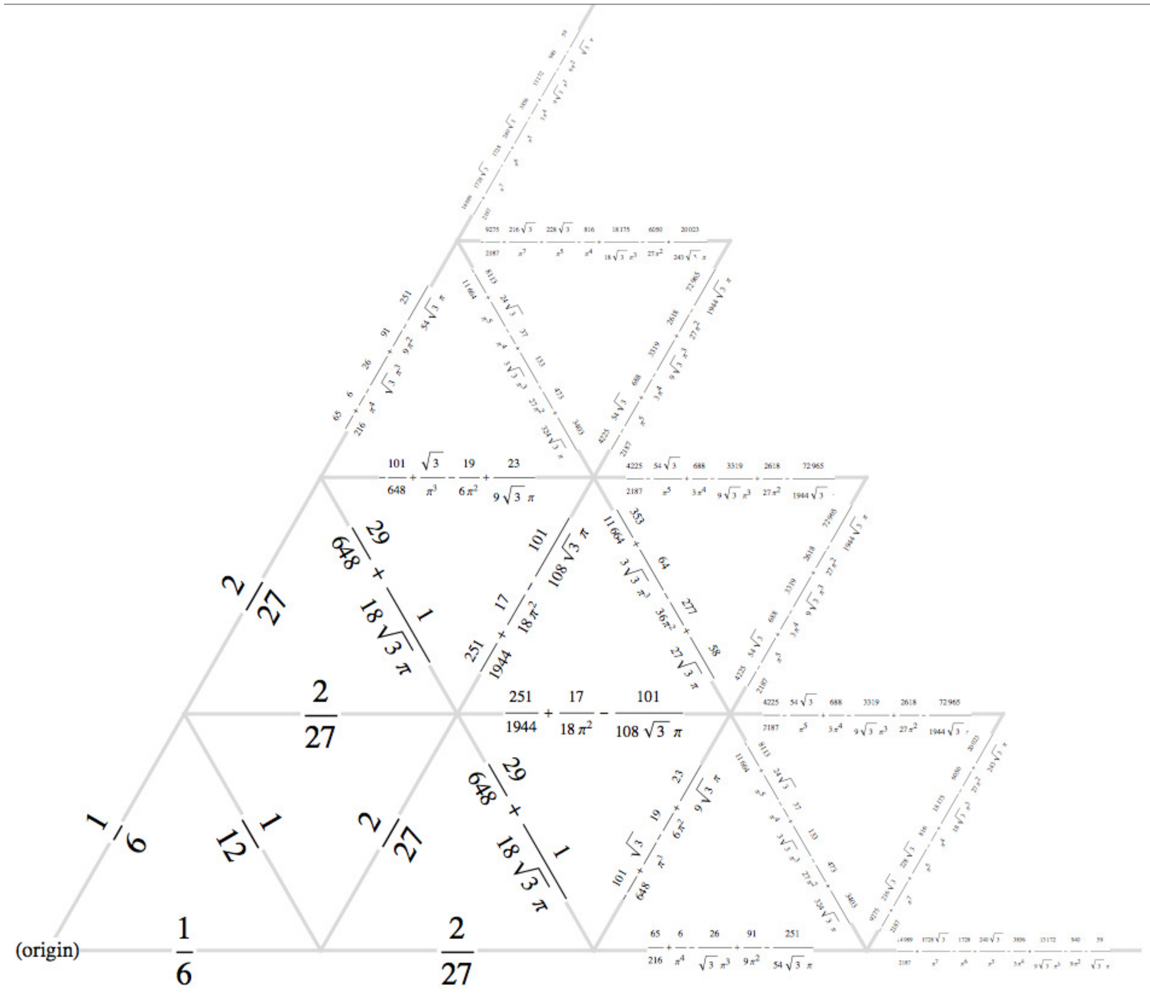$\frac{14989}{2187}+\frac{1728\sqrt{3}}{\pi^7}-\frac{1728}{\pi^6}-\frac{240\sqrt{3}}{\pi^5}-\frac{3856}{3\pi^4}+\frac{15172}{9\sqrt{3}\pi^3}-\frac{940}{9\pi^2}-\frac{59}{\sqrt{3}\pi}$
$\frac{9275}{2187}-\frac{216\sqrt{3}}{\pi^7}-\frac{228\sqrt{3}}{\pi^5}-\frac{816}{\pi^4}+\frac{18175}{18\sqrt{3}\pi^3}-\frac{6050}{27\pi^2}+\frac{20023}{243\sqrt{3}\pi}$
$\frac{65}{216}+\frac{6}{\pi^4}-\frac{26}{\sqrt{3}\pi^3}+\frac{91}{9\pi^2}-\frac{251}{54\sqrt{3}\pi}$
$\frac{8113}{11664}+\frac{24\sqrt{3}}{\pi^5}-\frac{37}{\pi^4}+\frac{133}{3\sqrt{3}\pi^3}-\frac{473}{27\pi^2}+\frac{3403}{324\sqrt{3}\pi}$
$\frac{4225}{2187}-\frac{54\sqrt{3}}{\pi^5}-\frac{688}{3\pi^4}+\frac{3319}{9\sqrt{3}\pi^3}-\frac{2618}{27\pi^2}+\frac{72965}{1944\sqrt{3}\pi}$
$\frac{101}{648}+\frac{\sqrt{3}}{\pi^3}-\frac{19}{6\pi^2}+\frac{23}{9\sqrt{3}\pi}$
$\frac{4225}{2187}-\frac{54\sqrt{3}}{\pi^5}-\frac{688}{3\pi^4}+\frac{3319}{9\sqrt{3}\pi^3}-\frac{2618}{27\pi^2}+\frac{72965}{1944\sqrt{3}\pi}$
$\frac{2}{27}$
$\frac{29}{648}+\frac{1}{18\sqrt{3}\pi}$
$\frac{251}{1944}+\frac{17}{18\pi^2}-\frac{101}{108\sqrt{3}\pi}$
$\frac{353}{11664}+\frac{64}{3\sqrt{3}\pi^3}-\frac{277}{36\pi^2}+\frac{58}{27\sqrt{3}\pi}$
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$\frac{1}{6}$
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$\frac{29}{648}+\frac{1}{18\sqrt{3}\pi}$
$\frac{101}{648}+\frac{\sqrt{3}}{\pi^3}-\frac{19}{6\pi^2}+\frac{23}{9\sqrt{3}\pi}$
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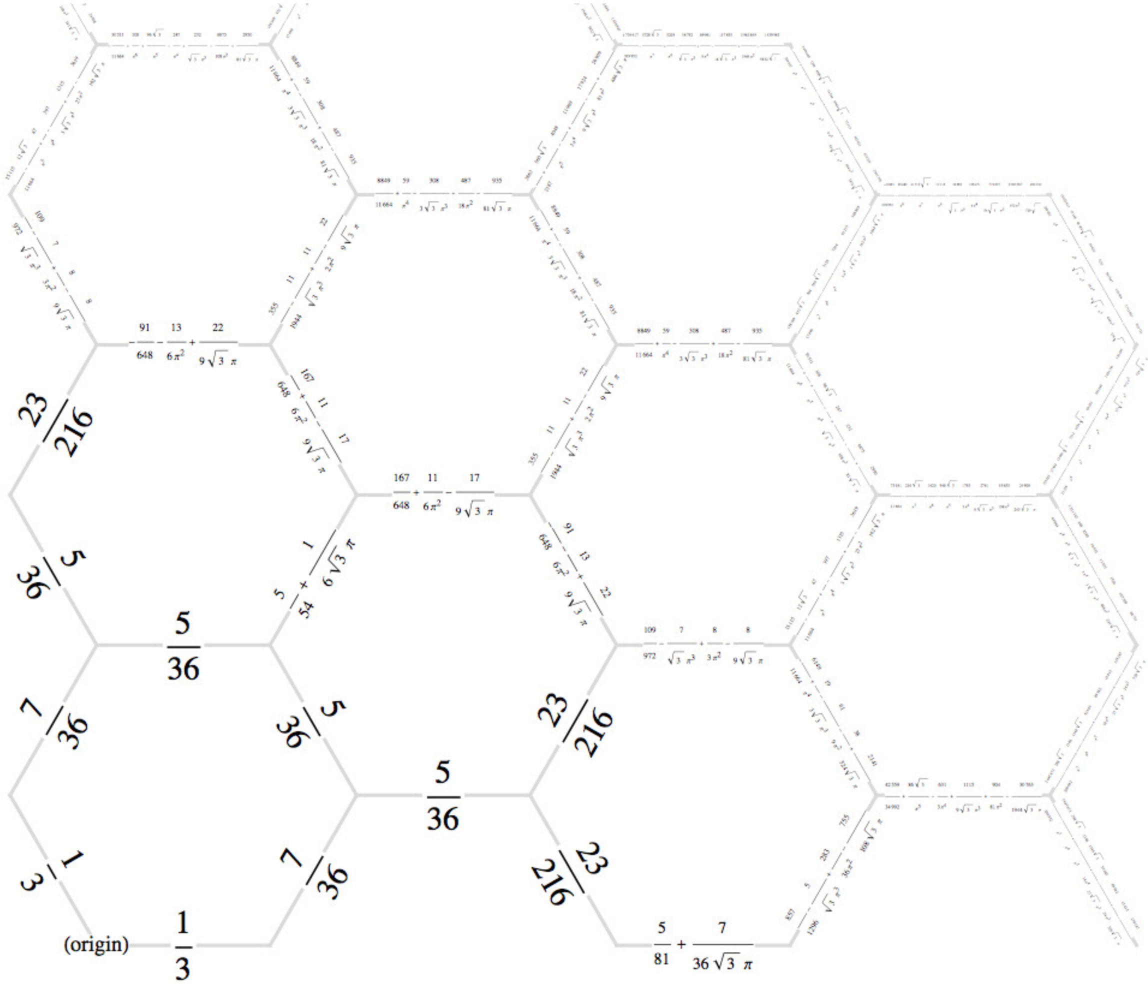
$\frac{65}{216}+\frac{6}{\pi^4}-\frac{26}{\sqrt{3}\pi^3}+\frac{91}{9\pi^2}-\frac{251}{54\sqrt{3}\pi}$
$\frac{14989}{2187}+\frac{1728\sqrt{3}}{\pi^7}-\frac{1728}{\pi^6}-\frac{240\sqrt{3}}{\pi^5}-\frac{3856}{3\pi^4}+\frac{15172}{9\sqrt{3}\pi^3}-\frac{940}{9\pi^2}-\frac{59}{\sqrt{3}\pi}$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$\frac{109}{972}-\frac{7}{\sqrt{3}\pi^3}+\frac{8}{3\pi^2}-\frac{8}{9\sqrt{3}\pi}$
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