- Thursday, May 03, 2012
- Speaker:Yuval Peres (joint work with Noam Berger)
- Title:Detecting the trail of the random walker
- Note taker:Xiaoqin Guo

1. Riddle (Kantorovich):Consider discrete time Markov chain on a space with n states.Can we bound (τ_y is the hitting time of y.
$$P(\exists y, \tau_y = t) \leq f(n,t) \stackrel{t\to\infty}{\longrightarrow} 0?$$
   - Example (Kozma, Zeitouni) on a graph of bounded degree, $f \sim n^2/t$.

2. On $\mathbb{Z}^d$, start with iid fair coins everywhere. Consider a random walk path starting from the origin (e.g. simple random walk, oriented random walk). On the random walk path S, replace the fair coin by $(q, 1-q)$ coins. This yields a law Q_S for the scenery. Define the average scenery
$$Q = \int Q_S \,\mathrm{d}\varphi(S),$$
where φ is the law on transient paths.
Question: Can we distinguish between the original scenery $P = \mu^{\mathbb{Z}^d}$ and Q? ($\mu = (\frac{1}{2}, \frac{1}{2})$ is the law of a fair coin. Denote $Q_S = \nu^S \mu^{\mathbb{Z}^d \setminus S}$. ν is the biased coin.)
For $d \leq 4$, we expect $P \perp Q$.

3. **Theorem** (Berger, P.)
(a) For SRW, $P \perp Q$ for all d and $q \neq \frac{1}{2}$.
(b) For oriented random walk, $P \perp Q$ for $d \leq 3$; $P \sim Q$ for $d \geq 4$ and some $q \neq \frac{1}{2}$.
(c) For $d = 2$, $q \neq \frac{1}{2}$, $P \perp Q$ for any φ.
For $d \geq 3$, $\exists q \neq \frac{1}{2}$ such that $P \sim Q$ for some φ.
   - Proposition: The SRW on $\mathbb{Z}^d, d \geq 3$ will for infinitely many n, fill more than half of a cube of side $(\log n)^{\frac{1}{d-2}}$ before reaching distance n from the origin.

4. Proof of (b), $d \geq 4$.
(Using techniques in Levin-Pemantle-P. and Bolthausen-Sznitman) We will prove the absolute continuity by bounding the l^2-norm of R-N derivative.
Lemma: $d \geq 4$, consider two independent copies of oriented random

walks $S, \tilde{S}$. $(\varphi \times \varphi)(|S \cap \tilde{S}| \geq k) \leq e^{-ck}$.
Look at a box of side n. We will show

$$\int (\frac{\mathrm{d}Q_n}{\mathrm{d}P_n})^2 \,\mathrm{d}P \leq B$$

for $B < \infty$ independent of n. (P_n, Q_n are measures restricted to the configuration ω of coins in a box of side n). Let Γ be the space of paths starting from the origin.

$$\begin{aligned}
\int (\frac{\mathrm{d}Q_n}{\mathrm{d}P_n})^2 \,\mathrm{d}P &= \int_\Omega \int_\Gamma \int_\Gamma \frac{Q_S(\omega) Q_{\tilde{S}}(\omega)}{P_n(\omega) P_n(\omega)} \,\mathrm{d}\varphi(S) \,\mathrm{d}\varphi(\tilde{S}) \,\mathrm{d}P_n \\
&= \int_\Omega \int_\Gamma \int_\Gamma \prod_{\substack{v \in \text{Box of side } n \\ \text{and } v \in S \cap \tilde{S}}} \frac{Q_S(\omega_v) Q_{\tilde{S}}(\omega_v)}{P(\omega_v)^2} \,\mathrm{d}\varphi(S) \,\mathrm{d}\varphi(\tilde{S}) \,\mathrm{d}P_n.
\end{aligned}$$

Note that $\int \frac{Q_S(\omega_v) Q_{\tilde{S}}(\omega_v)}{P(\omega_v)^2} = \int (\frac{\mathrm{d}\nu}{\mathrm{d}\mu})^2 \,\mathrm{d}\mu$. (b) follows by taking q near $\frac{1}{2}$.