- Thursday, May 03, 2012
- Speaker: Perla Sousi
- Title: Self-interacting random walks
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1. Model
   - μ_1, μ_2 are two zero-mean probability measures on $\mathbb{R}^3$. They are fully supported (i.e., Supp(μ_i) contain 3 linearly independent vectors).
   - $\xi_1^1, \ldots, \xi_i^1, \ldots$ are iid $\sim \mu_1$.
   - $\xi_1^2, \ldots, \xi_i^2, \ldots$ are iid $\sim \mu_2$.
   - Random walks $X_0 = 0$, $X_{t+1} = X_t + \xi_{t+1}^{l(t)}$, where $l(t) \in \{1,2\}$ is $\sigma(X_0, \ldots, X_t)$-measurable.

2. **Theorem 1**(Peres-Popov-S.) For any 2 such measures in $\mathbb{R}^d, d \geq 3$, and any adapted l, X_t is transient.

   Theorem 2(Peres-Popov-S.) In $\mathbb{R}^d$, there exist d such measures and an adapted rule l such that X_t is recurrent.
   (*Open question*: Transience for $d-1$ measures in $\mathbb{R}^d$?)

   - Proof for Theorem 1, the case $d \geq 5$: Use local CLT. Since

     $$\{X_n \in B(0,R)\} \subset \{\exists i_1, i_2 : i_1 + i_2 = n, \xi_1^1 + \ldots + \xi_{i_1}^1 + \xi_1^2 + \ldots + \xi_{i_2}^2 \in B(o,R)\},$$

     by LCLT,

     $$\begin{aligned} P(X_n \in B(0,R)) &\leq (n+1) \max_{i_1 > n/2} P(\xi_1^1 + \ldots + \xi_{i_1}^1 + \xi_1^2 + \ldots + \xi_{i_2}^2 \in B(o,R)) \\ &\leq C(n+1) R^d / n^{d/2}, \end{aligned}$$

     which is summable for $d \geq 5 \Longrightarrow$ transience.
   - Proof of Theorem 1, $d = 3$:
     Assume $Z_i \sim \mu_i$ and $\|Z_i\| \leq A < \infty$ a.-s., $i = 1,2$.
     Idea: Find a function $\varphi > 0$, $\varphi(x) \overset{|x| \to \infty}{\longrightarrow} 0$ such that $\varphi(X_t)$ is a supermartingale $\Rightarrow X_t$ is transient (by the martingale convergence theorem and $\overline{\lim} |X_t| = \infty$).
     Take $\varphi(x) = \|x\|^{-\alpha} \wedge 1$ for some $\alpha > 0$. It suffices to show

     $$E[\varphi(x + Z_i) - \varphi(x)] \leq 0, \quad \forall i = 1,2 \text{ and } x \text{ large}.$$

     Consider the Taylor expansion of φ ......

- Example of recurrence case: In $\mathbb{R}^3$, when the walker is at $x = (x_1, x_2, x_3)$, he picks the direction e_i such that $|x_i| = \max_j |x_j|$ with probability $1 - \epsilon$, and picks the other directions with probabilities $\epsilon/2$. Then add ± 1 to the chosen coordinate. When ϵ is small enough, the walk is recurrent.