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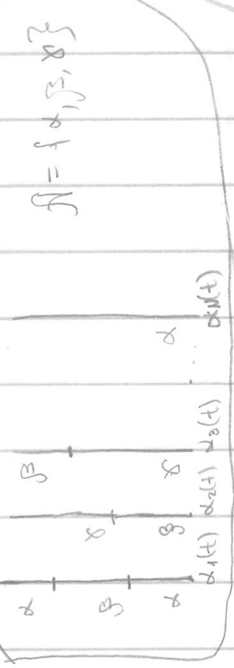
Stochastic Representation of the Ground States for the Mean-Field Models with a Transverse Component

(joint work with Dima Ioffe)

Let us begin with Stochastic Model and leave motivation for latter. Because in this case, the motivation is harder to understand then the Stochastic Model.



Stochastic Model:

- $\alpha(t)$ - continuous time Markov Chain on set $\mathcal{A}$ (with $|\mathcal{A}| = n$)
- $\lambda_{\alpha\beta}$ - jump rates. λ is irreducible ($\lambda \equiv 0$ on diagonal)
- $\mathbb{P}^N_{\underline{\alpha}}$ - the path measure for N independent copies of such chain starting from $\underline{\alpha}$.
- $m^N_\alpha(t) = \frac{1}{N} \# \{ i : \alpha_i(t) = \alpha \}$ $\alpha \in \mathcal{A}$
- $\underline{m}_N(t) = (m^N_{\alpha_1}(t), m^N_{\alpha_2}(t), \dots)$ $\alpha_i \in \mathcal{A}$ - mean process, continuous time Markov chain on simplex Δ^N_n

$(\underline{m} \in \Delta^N_n$ iff $\exists\ \underline{\alpha} \in \mathcal{A}^N : m_\alpha = \frac{1}{N} \# \{ i : \alpha_i = \alpha \}\ \forall \alpha \in \mathcal{A})$

$\Delta_n = \{ \underline{m} \in \mathbb{R}^n_+ : \sum_i m_i = 1 \}$ $\Delta^N_n = \Delta_n \cap \frac{1}{N} \mathbb{Z}^n$

Generator of the process $\underline{m}_N(t)$

$$G_N f(\underline{m}) = N \sum_{\alpha,\beta} m_\alpha \lambda_{\alpha\beta} \left(f\left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N}\right) - f(\underline{m}) \right).$$

$\underline{m}_N(t)$ is reversible with respect to the measure

$$\mu_N(\underline{m}) \stackrel{\Delta}{=} \frac{C_N(\underline{m})}{n^N}, \qquad C_N(\underline{m}) = \frac{N!}{\prod (N m_\alpha)!}$$

(2)

Let $\Psi_N(\underline{m})$ be such that $\Psi(\underline{m}) = \lim_{N\to\infty} \Psi_N(\underline{m}) = F\left(\frac{1}{2}\sum_{\alpha\beta} \rho_{\alpha\beta} m_\alpha m_\beta\right)$

(F-polynomial of finite degree , $\rho_{\alpha\beta}$ - "nice")

We are interested in the principal eigenfunction of $G_N + N\Psi(\underline{m})$ (denote by $g_N(\underline{m})$) with the corresponding top eigenvalue (denote by $-R_N$). And in the limits

$\lim_{N\to\infty} \frac{R_N}{N}$, $\lim_{N\to\infty} -\frac{1}{N}\log g_N(\underline{m})$ if exist

It will give us a description of the ground state

(We prove existence and uniqueness of $g_N(\underline{m}) > 0$ and R_N using Perron-Frobenius for $e^{-T\mathcal{H}_N}$ and using the stochastic representation.)

The corresponding eigenfunction equation is:

$$g_N(\underline{m}) = \mathbb{E}^N_{\underline{m}}\left[e^{N\int_0^T \Psi_N(\underline{m}(t))dt + TR_N} g_N(\underline{m}(T))\right].$$

$P_t = e^{tL}$

In order to find the limits above we have to take $T\to\infty$ first and then $N\to\infty$.
If we are alowed to exchange the limits then for fixed T as $N\to\infty$ one can prove the following theorem.

(Using LD for Markov Jump processes + Varadhan's Lemma)

(3)

Theorem: For $\underline{m}, \underline{m}' \in \Delta_n^N$ define

$$Z_T^N(\underline{m}', \underline{m}) \triangleq \frac{1}{N} \log \mathbb{E}_{\underline{m}'}^N \left[e^{N \int_0^T \Psi_N(\underline{m}(t))dt} \mathbb{1}_{\{\underline{m}(T)=\underline{m}\}} \right]$$

and

$$Z_T(\underline{m}', \underline{m}) \triangleq -\inf_{\substack{\gamma(0)=\underline{m}',\ \gamma(T)=\underline{m} \\ \gamma\text{-absolutely cont.}}} \int_0^T \mathcal{L}(\gamma(t), \gamma'(t))\, dt$$

where

$$\mathcal{L}(\underline{m}, \underline{v}) = \sup_{\underline{\theta}} \{ (\underline{v}, \underline{\theta}) - H(\underline{m}, \underline{\theta}) \} \quad \text{with}$$

$$H(\underline{m}, \underline{\theta}) = \sum_{\alpha,\beta} m_\alpha \lambda_{\alpha\beta} \left(e^{\theta_\beta - \theta_\alpha} - 1 \right) + \Psi(\underline{m})$$

(H is invariant under the shifts $\underline{\theta} \to \underline{\theta} + c\mathbb{1}$ => $\mathcal{L}$ is infinite whenever $(\underline{v}, \mathbb{1}) \neq 0$)

For all T sufficiently large the sequence of functions $\{Z_T^N\}$ is equi-continuous on $\Delta_n \times \Delta_n$ and uniformly locally Lipschitz on $\mathrm{int}(\Delta_n \times \Delta_n)$. Furthermore,

$$\lim_{N\to\infty} \max_{\underline{m}, \underline{m}'} | Z_T^N(\underline{m}', \underline{m}) - Z_T(\underline{m}', \underline{m}) | = 0$$

simultaneously for all $\underline{m}, m' \in \Delta_n$.

(Equi-continuity of $\{Z_T^N\}$ implies that the convergence is uniform. Consequently, $Z_T(\cdot,\cdot)$ is continuous on $\Delta_n \times \Delta_n$ and locally Lipschitz on $\mathrm{int}(\Delta_n \times \Delta_n)$.)

Back to the eigenfunction equation:

$$g_N(\underline{m}) = \mathbb{E}^N_{\underline{m}}\left[e^{N\int_0^T \Psi_N(\underline{m}(t))\,dt + TR_N} g_N(\underline{m}(T))\right].$$

By reversibility

$$\mu_N(\underline{m})\,\mathbb{E}^N_{\underline{m}}\left[e^{N\int_0^T \Psi_N(\underline{m}(t))\,dt} \mathbb{1}_{\{\underline{m}(T)=\underline{m}'\}}\right] = \mu_N(\underline{m}')\,\mathbb{E}^N_{\underline{m}'}\left[e^{N\int_0^T \Psi_N(\underline{m}(t))\,dt} \mathbb{1}_{\{\underline{m}(T)=\underline{m}\}}\right]$$

Define $\hat{g}_N(\underline{m}) = \mu_N(\underline{m})\, g_N(\underline{m})$ and $e^{-N\varphi_N(\underline{m})} \underset{\substack{\uparrow \\ g_N(\underline{m})>0}}{=} \hat{g}_N(\underline{m})$.

Then by reversibility

$$\hat{g}_N(\underline{m}) = \sum_{\underline{m}'} \hat{g}_N(\underline{m}')\,\mathbb{E}^N_{\underline{m}'}\left[e^{N\int_0^T \Psi_N(\underline{m}(t))\,dt + TR_N} \mathbb{1}_{\{\underline{m}(T)=\underline{m}\}}\right].$$

Using the Theorem above we can conclude that any accumulation point (r,φ) of the sequence $(\frac{1}{N}R_N, \varphi_N)$ satisfies: φ is locally Lipschitz on $\mathrm{int}(\Delta_n)$ and

$$\circledast\ \ \varphi(\underline{m}) = \inf_{\gamma(T)=\underline{m}} \left\{ \varphi(\gamma(0)) + \int_0^T \mathcal{L}(\gamma(t),\gamma'(t))\,dt \right\} - Tr \left(= \mathcal{U}_T\varphi(\underline{m}) - Tr \right)$$

$\forall T>0$ and each $\underline{m} \in \Delta_n$.

(The sequence $\frac{1}{N}R_N$ is bounded. The sequence $\{\varphi_N\}$ is equi-cont. on Δ_n and uniformly locally Lipschitz on $\mathrm{int}(\Delta_n)$)

(Accumulation points of φ_N are called admissible solutions of $\circledast$. Since
Lemma 5.10 $\mathcal{U}_T$ is non-expanding on $C(\Delta_n)$, validity of equation $\circledast$ unambiguously
Lemma 5.12 determines r, which implies that $r_1 \triangleq \lim_{N\to\infty} \frac{R_N}{N}$ exists.)

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Let's stop here and try to understand the motivation of the problem: connection to the ground states of the mean-field models with a transverse field and the results.

Class of Models:

- $\mathbb{X}$ - an n-dimensional complex Hilbert space
- $\{|\alpha\rangle\}_{\alpha\in\mathcal{A}}$ - orthonormal basis of $\mathbb{X}$
- $\mathcal{A}$ - set of classical labels $|\mathcal{A}| = n$ (the same set $\mathcal{A}$)
- $\mathbb{X}_N = \bigotimes_1^N \mathbb{X}$ with induced basis $|\underline{\alpha}\rangle = \bigotimes_{i=1}^N |\alpha_i\rangle$ $\alpha_i \in \mathcal{A}$.

The Hamiltonian

$$-\mathcal{H}_N = NF\left(\frac{1}{2N^2}\sum_{i\neq j} A_{ij}\right) + \sum_i B_i \quad \text{acts on } \mathbb{X}_N$$

- B_i-s - copies of Hermitian matrix B on $\mathbb{X}$. B_i acts on the i-th component of $|\underline{\alpha}\rangle$ only.
- $A_{ij} = A_{ji}$ - copies of Hermitian matrix A on $\mathbb{X}\otimes\mathbb{X}$. A_{ij} acts on i-th and j-th component of $|\underline{\alpha}\rangle$.

Assumptions:

1) A is diagonal: $A|\alpha,\beta\rangle = \rho_{\alpha\beta}|\alpha,\beta\rangle$
2) F - polynomial of finite degree
3) The transverse field B (non-diagonal) satisfies:
   $\forall \alpha,\beta\in\mathcal{A}$ $\lambda_{\alpha\beta} = \lambda_{\beta\alpha} \triangleq \langle\alpha|B|\beta\rangle \geq 0$
   λ is irreducible on $\mathcal{A}$.

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$$F\left(\frac{1}{2N^2}\sum_{i\neq j} A_{ij}\right)|\alpha\rangle = F\left(\frac{1}{2}\left(\sum_{\alpha\beta}\rho_{\alpha\beta} m_\alpha m_\beta - \frac{1}{N}\rho_{\alpha\beta} m_\alpha\right)\right)|\alpha\rangle \triangleq \Psi_N(\underline{m})|\alpha\rangle$$

$$\Psi(\underline{m}) = \lim_{N\to\infty} \Psi_N(\underline{m}) = F\left(\frac{1}{2}\sum_{\alpha\beta}\rho_{\alpha\beta} m_\alpha m_\beta\right)$$

$$m_\alpha = \frac{\#\{i : \alpha_i = \alpha\}}{N}$$

We are interested in the ground state of $\mathcal{H}_N$ - the eigenfunction corresponding to the bottom eigenvalue of $\mathcal{H}_N$.

For $\beta > 0$ the analogue of the Gibbs measure

$$\mu(\underline{\alpha}) = \frac{\langle \underline{\alpha} | e^{-\beta \mathcal{H}_N} | \underline{\alpha}\rangle}{\mathrm{Tr}\left(e^{-\beta\mathcal{H}_N}\right)}.$$

Set $\lambda = \sum_\alpha \sum_\beta \lambda_{\alpha\beta}$

The following stochastic representation of the entries of the density matrix holds: $\forall\, T \geq 0$ $\underline{\alpha}, \underline{\beta} \in \mathcal{A}^N$.

$$e^{-N\lambda T}\langle \underline{\beta} | e^{-T\mathcal{H}_N} | \underline{\alpha}\rangle = \mathbb{E}^N_{\underline{\alpha}}\left[e^{N\int_0^T \Psi_N(\underline{m}(t))dt}\, \mathbb{1}_{\{\underline{\alpha}(T) = \underline{\beta}\}}\right].$$

Define the mean vectors $|m\rangle \in \mathcal{X}_N$ $|\underline{m}\rangle \triangleq \frac{1}{\sqrt{c_N(\underline{m})}} \sum_{\underline{\alpha}\sim\underline{m}} |\underline{\alpha}\rangle$

Then

eigenfunction depends on $\underline{m}$ and not $\underline{\alpha}$ $\forall \underline{\alpha}, \underline{\alpha}'$: $m(\underline{\alpha}) = m(\underline{\alpha}')$

Lemma: If E_N is an eigenvalue of $\mathcal{H}_N$, then there exists a function h_N on Δ_n^N such that $|h_N\rangle \triangleq \sum_{\underline{m}\in\Delta_n^N} h_N(\underline{m})|\underline{m}\rangle$ is a corresponding eigenfunction:

$$\mathcal{H}_N|h_N\rangle = E_N|h_N\rangle.$$

$$e^{-N\lambda T}\langle \underline{m}' | e^{-T\mathcal{H}_N} | \underline{m}\rangle = \sqrt{\frac{\mu_N(\underline{m})}{\mu_N(\underline{m}')}}\, \mathbb{E}^N_{\underline{m}}\left[e^{N\int_0^T \Psi_N(\underline{m}(t))dt}\, \mathbb{1}_{\{\underline{m}(T) = \underline{m}'\}}\right].$$

Example: General spin-s model

- $n = 2s+1$ - dimension of X.
- $\mathcal{A} = \{-s, -s+1, \dots, s-1, s\}$ - (set of classical labels)
- The operators: $A_{ij} = S_i^z S_j^z$, $B_i = \lambda S_i^x$ (λ-strength of tr.f.)

wher,

$$S^z = \begin{pmatrix} s & & 0 \\ & s-1 & \\ 0 & & \ddots & -s \end{pmatrix} \qquad S^x = \frac{S^+ + S^-}{2} \qquad S^+ = \begin{pmatrix} 0 & C_s & 0 & \dots \\ 0 & 0 & C_{s-1} & 0 & \dots \\ \vdots & \vdots & & & 0 & C_{-s+1} \\ & & & & & 0 \end{pmatrix}$$

$$S^- = (S^+)^*, \quad C_m = \sqrt{s(s+1) - m(m-1)}.$$

$$-\mathcal{H}_N = NF\left(\frac{1}{2N^2}\sum_{i \neq j} S_i^z S_j^z\right) + \lambda \sum_i S_i^x$$

The operators S^x and S^z act on X as

- $S^z|\alpha\rangle = \alpha|\alpha\rangle$
- $S^x|\alpha\rangle = \frac{1}{2}\sqrt{s(s+1) - \alpha(\alpha-1)}\,|\alpha-1\rangle + \frac{1}{2}\sqrt{s(s+1) - \alpha(\alpha+1)}\,|\alpha+1\rangle$

=>

- $\rho_{\alpha\beta} = \alpha\beta$
- $\lambda_{\alpha\beta} = \begin{cases} \frac{\lambda}{2}\sqrt{s(s+1) - \alpha\beta}, & |\alpha - \beta| = 1 \\ 0, & \text{otherwise} \end{cases}$
- $\Psi(\underline{m}) = F\left(\frac{1}{2}\left(\sum_\alpha \alpha m_\alpha\right)^2\right)$

Spin-1/2 Model (Curie-Weiss in transverse field)

Take $\{-1,1\}$ instead of $\{-1/2, 1/2\}$ as a set of classical labels for spin-1/2 Model.

$$-\mathcal{H}_N = \frac{1}{2N}\sum_{i \neq j} \hat{\sigma}_i^z \hat{\sigma}_j^z + \lambda \sum_i \hat{\sigma}_i^x, \quad (F(t) = t)$$

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where $\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\hat{\sigma}^z |\alpha\rangle = \alpha |\alpha\rangle$, $\hat{\sigma}^x |\alpha\rangle = |-\alpha\rangle$ for $\alpha = \pm 1$; $\lambda_{-1 1} = \lambda_{1 -1} = \lambda$

Can be parametrized by a single variable

$m = m_1 - m_{-1} \in [-1, 1]$.

$\Psi_N(m) = \frac{1}{2} m^2$.

Results:

Theorem 1: Let E_N^1 be the bottom eigenvalue of $\mathcal{H}_N$. Then the limit

$$-\lambda + r_1 \triangleq \lim_{N \to \infty} \frac{E_N^1}{N}$$

exists. Moreover,

(1) r_1 is the solution of the minimization problem:

$$r_1 = \min_{\underline{m}} \mathcal{L}(\underline{m}, 0) = -\max_{\underline{m}} \min_{\underline{\theta}} H(\underline{m}, \underline{\theta}) = \min_{\underline{m}} \left\{ \frac{1}{2} \sum_{\alpha, \beta} \lambda_{\alpha\beta} \left(\sqrt{m_\alpha} - \sqrt{m_\beta}\right)^2 - \Psi(\underline{m}) \right\}.$$

(2) $\exists$ a continuous function φ on Δ_n : $\forall T \geq 0$, $\underline{m} \in \Delta_n$,

$$(*) \quad \varphi(\underline{m}) = \inf_{\substack{\gamma : \gamma(T) = \underline{m} \\ \gamma\text{-absolutely cont.}}} \left\{ \varphi(\gamma(0)) + \int_0^T \mathcal{L}(\gamma(t), \gamma'(t))\, dt - T r_1 \right\}$$

Theorem 2: The ground state of $\mathcal{H}_N$ is of the form

$$|h_N^1\rangle = \sum_{m \in \Delta_n^N} h_N^1(\underline{m}) |\underline{m}\rangle, \quad \left[|\underline{m}\rangle \triangleq \frac{1}{\sqrt{C_N(\underline{m})}} \sum_{\underline{\alpha} \sim \underline{m}} |\underline{\alpha}\rangle \right.$$

Assume $\exists$ a unique ~~admissible~~ solution φ^* of $(*)$. Let $\mathrm{ent}(\underline{m}) \triangleq -\sum m_\alpha \log m_\alpha$

Set $\psi^*(\underline{m}) = 2\varphi^*(\underline{m}) + \mathrm{ent}(\underline{m}) - 2\log n$. Then uniformly in $\underline{m} \in \Delta_n^N$,

$$-1/N \log h_N^1(\underline{m}) = \frac{1}{2} \psi^*(\underline{m}) + O_N(1)$$

(where $O_N(1) \to 0$ as $N \to \infty$).

For the Curie-Weiss Model in transverse field:

- if $\lambda > 1$ $\mu_1 = 0$
- if $\lambda \leq 1$ $\mu_1 = -\frac{(\lambda-1)^2}{2}$ and

$$\psi^*(\underline{m}) = \left((1-|\underline{m}|)\log\frac{1-|m|}{\lambda} + |m|\right) + \text{ent}(m) + 2\log 2.$$

Stochastic Representation of the Ground State.

The eigenfunction equation defines a Markovian semi-group.

$$\hat{\mathbb{E}}_T^N f(\underline{m}) = \frac{1}{g_N^1(\underline{m})}\,\mathbb{E}_{\underline{m}}^N\left[e^{N\int_0^T \Psi_N(\underline{m}(t))dt + TR_N^1}\, g_N^1(\underline{m}(T))\, f(\underline{m}(T))\right]$$

This corresponds to continuous time Markov chain with the generator

$$\hat{G}_N f(m) = \frac{N}{g_N^1(\underline{m})}\sum_{\alpha,\beta} m_\alpha \lambda_{\alpha\beta}\, g_N^1\left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N}\right)\left(f\left(\underline{m} + \frac{\delta_\beta - \delta_\alpha}{N}\right) - f(\underline{m})\right)$$

We call $\hat{G}_N$ the generator of the ground state chain

One can prove:

The generator $\hat{G}_N$ is reversible with respect to the measure $\nu_N(\underline{m}) \triangleq (h_N^1(\underline{m}))^2$. Furthermore, E_N is an eigenvalue of $\mathcal{H}_N$ iff $E_N^1 - E_N$ is an eigenvalue of $\hat{G}_N$.