- Speaker:Balint Toth (joint work with Horvath and Veto)
- Title:Scaling limits for self-interacting random walks and diffusions
- Note-taker:Xiaoqin Guo

1. Model. Consider continuous time nearest neighbor jump process $t \mapsto X(t) \in \mathbb{Z}^d$. Let

$$l(t,z) = l(0,z) + |\{s \le t : X(s) = z\}|.$$

$w : \mathbb{R} \to (0,\infty)$ is an increasing function. Define the law of the (self-repelling) walks:

$$P(X(t+\,\mathrm{d}t) = y|\mathcal{F}_t, X(t) = x) = 1_{x\sim y} w(l(t,x) - l(t,y))\,\mathrm{d}t + o(\,\mathrm{d}t).$$

Define

$$s(u) = \frac{w(u)+w(-u)}{2}, r(u) = \frac{w(u)-w(-u)}{2}.$$

Assume ellipticity: $\inf_u w(u) = \gamma > 0$.

2. Conjecture (Amit-Parisi-Peliti):
$d=1$, $X(t) \sim t^{2/3}$, "-strange" limit
$d=2$, $X(t) \sim t^{1/2}(\log t)^{1/4}$, "Gaussian"
$d \ge 3$, $X(t) \sim t^{1/2}$, Gaussian behavior.

Related works:

- $d=1$, B.Toth '95(related model), Toth+Werner '98, Toth+Veto '10, Tarres+Toth+Valko '11.
- $d=2$,$EX(t)^2 \gg t\log\log t$, Toth+Valko '11.
- $d \ge 3$, this talk.

3. Environment viewed from the random walker: $\eta(t) = (\eta(t,x))_{x\in\mathbb{Z}^d} := (l(t,X(t)+x))_{x\in\mathbb{Z}^d}$ is a cadlag Markov process in

$$\Omega = \left\{\omega = (\omega(x))_{x\in\mathbb{Z}^d} : \lim_{|x|\to\infty} \frac{|\omega(x)|}{|x|^\epsilon} = 0, \forall\epsilon > 0\right\}.$$

τ_z: translation from $\Omega \to \Omega$, $\mathcal{U} := \{e : |e| = 1\}$.
$(\eta(t))$ has generator

$$(Gf)(\omega) = \sum_{e\in\mathcal{U}} w(\omega(o) - \omega(e))(f(\tau_e\omega) - f(\omega)) + Df(\omega),$$

where $Df(\omega) = \frac{\partial f}{\partial\omega(0)}(\omega)$.

4. The stationary and ergodic measure for the environment process $\eta(t)$.
Define the Gibbs measure π on Ω: for $\Lambda \subset \mathbb{Z}^d$,

$$\mathrm{d}\pi(\omega_x|\omega_{\Lambda^c}) = Z_\Lambda^{-1} \exp\{-\frac{1}{2} \sum_{\substack{x \sim y \\ x,y \in \Lambda}} R(w(x)-w(y)) - \sum_{\substack{x \sim y, \\ x \in \Lambda, y \in \Lambda^c}} R(\omega(x)-\omega(y))\} \prod_{x \in \Lambda} \mathrm{d}\omega_x,$$

where $R : \mathbb{R} \to [0, \infty)$ is defined by $R(u) = \int_0^u r(v)\,\mathrm{d}v$.
Remark: If $R(u) = u$, then π is the free Gaussian field on $\mathbb{Z}^d$.

5. Theorem:

   1) Under minimal technical conditions on $w(\cdot)$,

   $$\gamma \le \overline{\lim_{t \to \infty}} \frac{1}{t} E[X(t)^2] < \infty.$$

   2) Assume $r(u) = u$, $s(u) = au^4 + bu^2 + c$ (a, b, c are constants chosen s.t. $s > 0$). Then

   $$\lim_{t \to \infty} \left(\frac{E[X_k(t)X_l(t)]}{t} \right) = (a_{kl}) > 0,$$

   and the finite dimensional marginal distribution of $\frac{X(Nt)}{\sqrt{N}} \Longrightarrow$ Brownian motion with diffusion matrix (a_{kl}).

6. Proof: Split $X(t)$ as

$$X(t) = (X(t) - \int_0^t \varphi(\eta_s)\,\mathrm{d}s) + \int_0^t \varphi(\eta_s)\,\mathrm{d}s$$

such that $(X(t) - \int_0^t \varphi(\eta_s)\,\mathrm{d}s)$ is a martingale. Then use Kipnis-Varadhan theory to control $\int_0^t \varphi(\eta_s)\,\mathrm{d}s$.