Excited random walks on $\mathbb{Z}^d$

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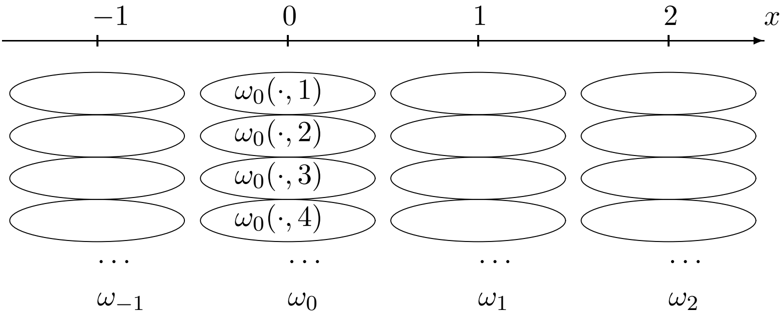
Model description: cookie environments

- Let $\mathcal{E} := \{\pm e_j \mid j \in \{1, 2, \dots, d\}\}$ be the set of unit coordinate vectors in $\mathbb{Z}^d$ and denote by $\mathcal{M}_{\mathcal{E}}$ the set of probability measures on $\mathcal{E}$, each of which is called “a cookie”. The set of cookie environments is denoted by $\Omega := \mathcal{M}_{\mathcal{E}}^{\mathbb{Z}^d \times \mathbb{N}}$.



Dynamics of excited random walk

Let $\mathbb{P}$ be a probability measure on Ω. Denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$.

- Random walk under the quenched measure: for $\omega \in \Omega$ let $P_{0,\omega}$ be a probability measure on the set of nearest neighbor paths such that $P_{0,\omega}(X_0 = 0) = 1$ and

$$P_{0,\omega}(X_{n+1} = X_n + e \,|\, (X_m)_{m \le n}) = \omega_{X_n}(e, L_{X_n}(n)), \;\; e \in \mathcal{E},$$

  where $L_z(n) := \sum_{m=0}^{n} 1_{\{X_m = z\}}$ be the number of visits to z up to time n.
- The averaged measure for $X := (X_n)_{n \ge 0}$ is defined as follows: $P_0(\cdot) := \mathbb{E}(P_{0,\omega}(\cdot))$.

Assumptions on the environment

We shall assume that either

(IID) $\omega_z,\ z \in \mathbb{Z}^d,$ are i.i.d. under $\mathbb{P}$ or

(SE) $\omega_z,\ z \in \mathbb{Z}^d,$ are stationary, ergodic w.r.t. to the shifts on $\mathbb{Z}^d$.

Moreover, one of the following ellipticity conditions will be in force:

(WEL) $\forall z \in \mathbb{Z}^d, e \in \mathcal{E}$: $\mathbb{P}\left[\forall i \in \mathbb{N} :\ \omega_z(e,i) > 0\right] > 0.$

(EL) $\forall z \in \mathbb{Z}^d, e \in \mathcal{E}$ and $i \in \mathbb{N} :$ $\mathbb{P}$-a.s. $\omega_z(e,i) > 0.$

(UEL) $\exists \kappa > 0 :\ \forall z \in \mathbb{Z}^d, e \in \mathcal{E}, i \in \mathbb{N}\ \ \omega_z(e,i) \geq \kappa$ $\mathbb{P}$-a.s.

Obviously, (UEL)$\Rightarrow$(EL)$\Rightarrow$(WEL).

Sometimes we shall assume that $\exists\,\ell\in\mathbb{R}^d\setminus\{0\}$ such that

$$\text{(POS}_\ell\text{)}\quad \sum_{e\in\mathcal{E}}\omega_z(e,i)\,e\cdot\ell\ge 0\quad \mathbb{P}\text{-a.s. }\forall i\in\mathbb{N},\ \forall z\in\mathbb{Z}^d.$$

The (possibly infinite) number of biased cookies at site z is denoted by

$$M(\omega_z):=\inf\{j\in\mathbb{N}_0\mid\forall e\in\mathcal{E}\ \forall i>j:\omega_z(e,i)=1/(2d)\}.$$

General properties: finite of infinite range

For $z \in \mathbb{Z}^d$ and $e \in \mathcal{E}$ write $z \stackrel{\omega}{\to} z+e$ if and only if $\sum_{i\geq 1} \omega_z(e,i) = \infty$. Define $b_F := \mathbb{P}[\forall e \in F : 0 \stackrel{\omega}{\not\to} e]$ for $F \subseteq \mathcal{E}$. The transitive closure in $\mathbb{Z}^d$ of the relation $\stackrel{\omega}{\to}$ is denoted also by $\stackrel{\omega}{\to}$.

Lemma

Let $\omega \in \Omega$ and $x, y \in \mathbb{Z}^d$ with $x \stackrel{\omega}{\to} y$. Then on the event that the ERW visits x infinitely often, y is $P_{0,\omega}$-a.s. visited infinitely often as well.

Theorem

Assume (IID) and (EL). If there is an orthogonal set $F \subset \mathcal{E}$ such that $b_F = 0$ then the range is P_0-a.s. infinite. If there is no such set then the range is P_0-a.s. finite.[1]

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[1] Kosygina, Zerner (2012, arxiv), used a lemma from Holmes, Salisbury (2011, arxiv)

General properties: recurrence and transience

Theorem ($d=1$)

(a) Assume (SE) and (EL). Then the ERW is either recurrent or transient or has P_0-a.s. finite range.
(b) Assume (SE), (WEL) and $\mathbb{P}$-a.s. $M(\omega_0) \leq 1$. Then ERW is recurrent.

Theorem ($d \geq 1$, Kalikow-type zero-one law[2])

Assume (IID) and (EL). For $\ell \in \mathbb{R}^d \setminus \{0\}$ define $A_\ell := \{\lim_{n\to\infty} X_n \cdot \ell = \infty\}$. Then $P_0[|X_n \cdot \ell| \to \infty] = P_0[A_\ell \cup A_{-\ell}] \in \{0,1\}$.

Theorem ($d \geq 2$ directional transience[3])

Assume (IID), (UEL), and (POS$_\ell$) for some $\ell \in \mathbb{R}^d \setminus \{0\}$. If $\mathbb{E}\left[\sum_{i\geq 1, e\in\mathcal{E}} \omega_0(e,i) e \cdot \ell\right] > 0$, then $P_0(A_\ell) = 1$.

[2] Kosygina, Zerner (2012, arxiv)
[3] Zerner (2006)

Open problems

(1) Let $d \geq 2$. Find conditions, which imply the zero-one law $P_0[A_\ell] \in \{0,1\}$ for all $\ell \in \mathbb{R}^d \setminus \{0\}$.
(2) Assume (IID) and (UEL) and suppose that ERW is balanced: $\omega_z(e,i) = \omega_z(-e,i)$ for all $z \in \mathbb{Z},\ i \in \mathbb{N},\ e \in \mathcal{E}$. Is it true that such walk is recurrent in $d = 2$ and transient for $d \geq 3$? For RWRE this is true[4].
(3) A non-elliptic version of this problem[5]: Let $d = d_1 + d_2$ and suppose that upon the first visit to to a vertex the walker performs a d_1-dimensional SSRW step in the first d_1 coordinates but upon subsequent visits to the same vertex he makes a SSRW step in the last d_2 coordinates. The authors of the problem gave a proof of transience when $d_1 = d_2 = 2$.

[4] see Zeitouni, LNM 1837 (2004), Th.3.3.22
[5] Benjamini, Kozma, Schapira (2011)

Regeneration structure[6]

FLT

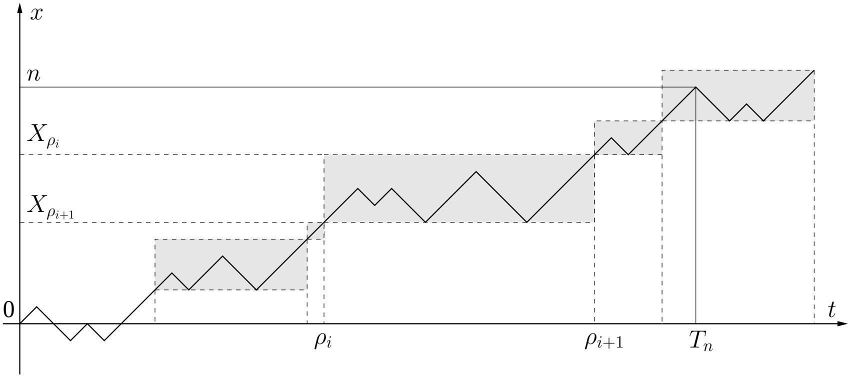

Figure: Regeneration structure for $d = 1$: sizes and contents of the shaded boxes are i.i.d..

[6] goes back to H. Kesten, M.V. Kozlov, F. Spitzer (1975); H. Kesten (1977)

Lemma (Existence of regeneration structure[7])

Assume (IID) *as well as* (WEL) *if* $d = 1$ *and* (EL) *if* $d \geq 2$. *Let* $\ell \in \mathbb{R}^d \backslash \{0\}$ *satisfy* $P_0[A_\ell] > 0$.
Then there are $P_0[\,\cdot \mid A_\ell]$*-a.s. infinitely many random times* $\tau_1 < \tau_2 < \dots$, *so-called* regeneration times, *such that*

$$X_m \cdot \ell < X_{\tau_k} \cdot \ell \;\; \forall m < \tau_k \;\; \text{and} \;\; X_m \cdot \ell \geq X_{\tau_k} \cdot \ell \;\; \forall m \geq \tau_k, \;\; k \in \mathbb{N},$$

the random $\bigcup_{n \in \mathbb{N}} (\mathbb{Z}^d)^n$*-valued vectors*

$$(X_n)_{0 \leq n \leq \tau_1}, \; (X_n - X_{\tau_i})_{\tau_i \leq n \leq \tau_{i+1}} \; (i \geq 1)$$

are independent w.r.t. $P_0[\,\cdot \mid A_\ell]$. *Moreover, the vectors* $(X_n - X_{\tau_i})_{\tau_i \leq n \leq \tau_{i+1}} \; (i \geq 1)$ *have the same distribution under* $P_0[\,\cdot \mid A_\ell]$ *as* $(X_n)_{0 \leq n \leq \tau_1}$ *under* $P_0[\,\cdot \mid \forall n \; X_n \cdot \ell \geq 0]$. *Also* $E_0[(X_{\tau_2} - X_{\tau_1}) \cdot \ell \mid A_\ell] < \infty$.

[7] Zerner (2006), Berard, Ramirez (2007)

Theorem (**directional law of large numbers)**

Under the assumptions of the above lemma the following holds: P_0-a.s. on the

$$\lim_{n\to\infty} \frac{X_n \cdot \ell}{n} = v_\ell := \frac{E_0[(X_{\tau_2} - X_{\tau_1}) \cdot \ell \mid A_\ell]}{E_0[\tau_2 - \tau_1 \mid A_\ell]} \in [0,1].$$

Theorem (**law of large numbers)**

Assume (IID) *as well as* (WEL) *if* $d = 1$ *and* (EL) *if* $d \geq 2$*. Let* $\ell_1, \dots, \ell_d$ *be a basis of* $\mathbb{R}^d$ *such that* $P_0[A_{\ell_i} \cup A_{-\ell_i}] = 1$ *for all* $i = 1, \dots, d$*. Then there are* $v \in \mathbb{R}^d$ *and* $c \geq 0$ *such that* P_0*-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} \in \{v, -cv\}.$$

If, in addition, $P_0[A_{\ell_i}] = 1$ *for all* $i = 1, \dots, d$ *then* X *satisfies a strong law of large numbers with velocity* $v \in \mathbb{R}^d$ *such that* $v \cdot \ell_i \geq 0$ *for all* $i = 1, \dots, d$*. (Proof follows Drewitz, Ramirez (2010).)*

Open problem

Assume (IID), (UEL), and (POS$_\ell$) for some $\ell \in \mathbb{R}^d$. Suppose also that $\mathbb{E}\left[\sum_{i\geq 1, e\in\mathcal{E}} \omega_0(e,i) e\cdot \ell\right] > 0$. Under these conditions ERW is transient in the direction ℓ. When is it ballistic?

No excitation after the first visit

The original ERW model and a modification[8].
Let $p \in (1/2, 1]$ and $\mathbb{P} = \delta_\omega$, where $\forall z \in \mathbb{Z}^d$,

(BW)
$$\omega(z, e_1, 1) = \frac{p}{d}, \quad \omega(z, -e_1, 1) = \frac{1-p}{d}, \text{ and}$$
$$\omega(z, e, i) = \frac{1}{2d} \quad \text{if } i \in \mathbb{N} \text{ and } e \in \mathcal{E} \setminus \{e_1, -e_1\} \text{ or if } i \geq 2.$$

Generalization: Assume that $\mathbb{P}$ satisfies (UEL), (IID), and that for some $\ell \in \mathbb{R}^d \setminus \{0\}$

(MPRV$_\ell$)
$$\exists \lambda > 0 : \sum_{e \in \mathcal{E}} \omega(0, e, 1)\, e \cdot \ell \geq \lambda \quad \mathbb{P}\text{-a.s. and}$$
$$\omega(0, e, i) = \omega(0, -e, i)) \ \ \mathbb{P}\text{-a.s. for all } i \geq 2, \ e \in \mathcal{E}.$$

[8] Benjamini, Wilson (2003), Menshikov, Popov, Ramirez, Vachkovskaia (2012)

ERW is called ballistic if it satisfies a SLLN with non-zero speed.

Theorem ($d \geq 2$, ballisticity and FCLT[9])

Let $\ell \in \mathbb{R}^d \backslash \{0\}$ and assume (MPRV_ℓ), (IID) and (UEL). Then the ERW is ballistic and its velocity v satisfies $v \cdot \ell > 0$.
Moreover, there exists a non-degenerate $d \times d$ matrix G such that with respect to P_0,

$$\frac{X_{[n\cdot]} - [n\cdot]v}{\sqrt{n}} \overset{J_1}{\Rightarrow} B_G(\cdot) \quad \text{as } n \to \infty,$$

where B_G is the d-dimensional Brownian motion with covariance matrix G.

Open question: under which conditions on d, the “strength” of the first cookie, and the underlying process, the first cookie determines the direction of the velocity? (See Holmes (2012).)

[9] Berard, Ramirez (2007), Menshikov, Popov, Ramirez, Vachkovskaia (2012)

Boundedly many positive and negative cookies per site, $d = 1$

Assume (IID), (WEL), and

$$\text{(BD)} \quad \exists M \in \mathbb{N}\text{: } \mathbb{P}\text{-a.s. } M(\omega_0) \le M.$$

The approach is based on the study of local times and analogs of Ray-Knight theorems.[10]
Continuous space-time analog, excited Brownian motions, was introduced and studied by Raimond, Schapira (2011).

[10] Harris (1952), Knight (1963), Kesten, Kozlov, Spitzer (1975), Toth (1996).

Basic properties[11]

- **Recurrence and transience:**

$0 \le \delta \le 1$ X is recurrent, i.e. for $\mathbb{P}$-a.a. environments ω, X returns $P_{0,\omega}$-a.s. infinitely many times to its starting point.

$\delta > 1$ X is transient to the right, i.e. for $\mathbb{P}$-a.a. environments ω, $X_n \to \infty$ as $n \to \infty$ $P_{0,\omega}$-a.s..

[11] Zerner (2005), Mountford, Pimentel, Valle (2006), Basdevant, Singh (2008); Kosygina, Zerner (2008)

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- **Strong law of large numbers:**
There is a deterministic $v \in [0,1]$ such that for $\mathbb{P}$-a.a. environments ω, $\lim_{n\to\infty} \frac{X_n}{n} = v \;\; P_{0,\omega}$-a.s..

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- **Ballisticity:**
  Let v be the (linear) speed defined above. Then $v > 0$ iff $\delta > 2$.

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- **Ballisticity:**
Let v be the (linear) speed defined above. Then $v > 0$ iff $\delta > 2$. Moreover, in the transient case with $v = 0$ we have

$1 < \delta < 2$ $\frac{X_n}{n^{\delta/2}} \Rightarrow$ a power of a strictly stable r. v. with index $\delta/2$;

$\delta = 2$ $\frac{X_n}{n/\log n} \overset{\text{prob.}}{\to} c \in (0,\infty)$.

[11] Zerner (2005), Mountford, Pimentel, Valle (2006), Basdevant, Singh (2008); Kosygina, Zerner (2008)

Transient case: centering and scaling of X_n and T_n.

	$\xi_n(t)$	$\eta_n(t)$
$\delta > 4$	$\frac{X_{[nt]} - [nt]v}{v^{3/2}\sqrt{n}}$	$\frac{T_{[nt]} - [nt]v^{-1}}{\sqrt{n}}$
$\delta = 4$	$\frac{X_{[nt]} - [nt]v}{v^{3/2}\sqrt{n \log n}}$	$\frac{T_{[nt]} - [nt]v^{-1}}{\sqrt{n \log n}}$
$2 < \delta < 4$	$\frac{X_{[nt]} - [nt]v}{v^{1+2/\delta} n^{2/\delta}}$	$\frac{T_{[nt]} - [nt]v^{-1}}{n^{2/\delta}}$

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$\delta = 2$	$\frac{X_{[nt]}-c[nt]\Gamma([n])}{c^2n(\log n)^{-2}}$	$\frac{T_{[nt]}-c^{-1}[nt]D([n])}{n}$
$1 < \delta < 2$	$\frac{X_{[nt]}}{n^{\delta/2}}$	$\frac{T_{[nt]}}{n^{2/\delta}}$

Here $t \geq 0$, $\Gamma(n) \sim 1/\log n$, $D(n) \sim \log n$ as $n \to \infty$. FLT

Convergence of one-dimensional distributions[12]

For $\alpha \in (0,2]$, $b>0$ let $Z_{\alpha,b}$ be a strictly stable random variable with index α, skewness $\beta=1$, and scale b, i.e. $(\operatorname{sign} 0 = 0)$

$$\log Ee^{iuZ_{\alpha,b}} = \begin{cases} -b^\alpha |u|^\alpha (1 - i(\operatorname{sign} u)\tan\frac{\pi\alpha}{2}) & \text{if } \alpha \neq 1; \\ -b|u|(1+\frac{2i}{\pi}(\operatorname{sign} u)\ \log|u|) & \text{if } \alpha = 1. \end{cases}$$

[12] Basdevant, Singh (2008); Kosygina, Zerner (2008); Kosygina, Mountford (2011)

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Let $X_n^* = \max_{0\le m\le n} X_m$. Then $\{T_n\le k\}=\{X_k^*\ge n\}$. This allowed us to transfer results from T_n to X_n (provided that we can control $\inf_{m\ge n} X_m$).

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Theorem

$$\eta_n(1) \Rightarrow \begin{cases} Z_{\delta/2,b}, & \text{if } \delta \in (1,4); \\ Z_{2,b} \sim N(0,2b^2), & \text{if } \delta \ge 4. \end{cases}$$

Constant $b>0$ depends on the cookie distribution. This can be translated into a result about $\xi_n(1)$.

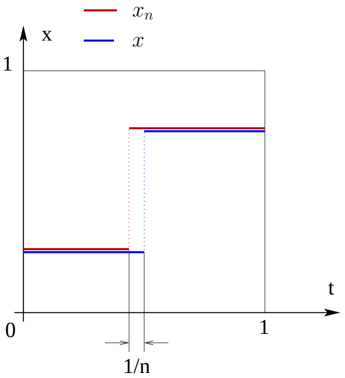
[12] Basdevant, Singh (2008); Kosygina, Zerner (2008); Kosygina, Mountford (2011)

Two Skorohod topologies, J_1 versus M_1

Consider $D([0,1])$. Recall that for $x, y \in D([0,1])$

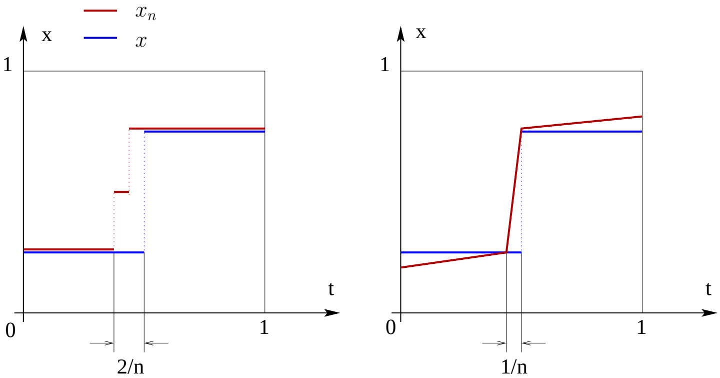
$$\rho_{J_1}(x,y) = \inf_{\lambda \in \Lambda} \Big[\sup_{t \in [0,1]} |x_n(t) - x(\lambda(t))| + \sup_{t \in [0,1]} |\lambda(t) - t| \Big]$$

For each of the two pictures below $\rho_{J_1}(x_n, x) \to 0$ as $n \to \infty$.



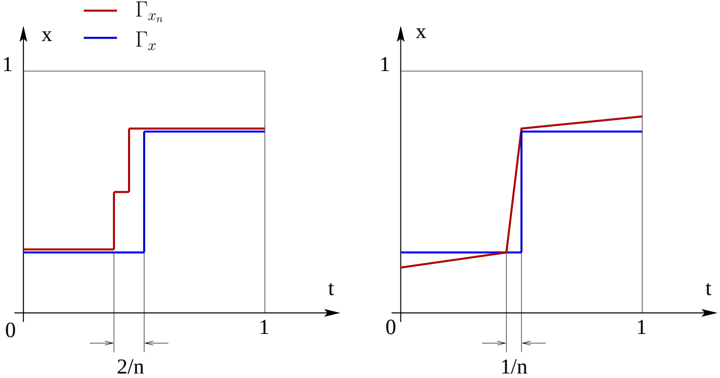
Two Skorohod topologies, J_1 versus M_1

M_1 topology is weaker than J_1. Informally (and with some omissions), $\rho_{M_1}(x,y) =$" distance between the completed graphs of x and y, Γ_x and Γ_y". For each of the two pictures below $\rho_{M_1}(x_n,x) \to 0$ as $n \to \infty$ but $\rho_{J_1}(x_n,x) \not\to 0$.



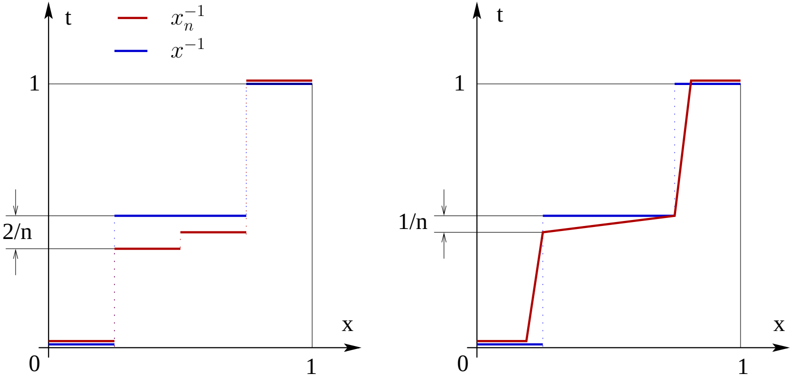
Two Skorohod topologies, J_1 versus M_1

M_1 topology is less demanding than J_1. Informally (and with some omissions), $\rho_{M_1}(x, y) =$" distance between the completed graphs of x and y, Γ_x and Γ_y". For each of the two pictures below $\rho_{M_1}(x_n, x) \to 0$ as $n \to \infty$ but $\rho_{J_1}(x_n, x) \not\to 0$.



Two Skorohod topologies, J_1 versus M_1

Consider the inverse maps and notice that for the first picture $\rho_U(x_n^{-1}, x^{-1}) \to 0$ and for the second $\rho_{M_1}(x_n^{-1}, x^{-1}) \to 0$ as $n \to \infty$.



Functional limit theorems

Recall IID structure

Theorem

If $\delta \in (1,2)$ *then* $\xi_n \overset{J_1}{\Rightarrow} \xi$ *as* $n \to \infty$. *Here* ξ *is the inverse of the stable subordinator* η *for which* $\eta(1) \overset{\mathrm{d}}{=} Z_{\delta/2,b}$.
If $\delta \in (2,4)$ *then*

$$\xi_n \overset{M_1}{\Rightarrow} \xi \quad \text{as} \quad n \to \infty,$$

where ξ *is a stable process with index* $\delta/2$ *such that* $\xi(1) \overset{d}{=} -Z_{\delta/2,b}$. *Moreover,* $\xi_n \overset{J_1}{\not\Rightarrow} \xi$.
If $\delta \geq 4$ *then*

$$\xi_n \overset{J_1}{\Rightarrow} \sqrt{2b}\, B \quad \text{as} \quad n \to \infty,$$

where B *is the standard Brownian motion*[13].

[13] Kosygina, Zerner (2012,arxiv)

Recurrent case

- If $0 \le \delta < 1$ then $\xi_n(t) = \frac{X_{[nt]}}{\sqrt{n}}$ converges to Brownian motion perturbed at extrema[14]. More precisely,

Theorem

Assume that $0 \le \delta < 1$. Let $W(\cdot)$ be the unique pathwise solution[15] of

$$W(t) = B(t) + \delta(\max_{0\le s\le t} W(s) - \min_{0\le s\le t} W(s)), \quad W(0) = 0,$$

where B is the standard Brownian motion. Then $\xi_n \overset{J_1}{\Rightarrow} W$.

- The process W is not defined for $\delta = 1$. What is the limiting process for the case $\delta = 1$?

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[14] Dolgopyat (2011), Dolgopyat, Kosygina (2012, arxiv)

[15] Carmona, Petit, Yor (1998), Chaumont, Doney (1999)

Boundary case $\delta = 1$[16]

Theorem

Let $\delta = 1$. *Then*

$$\frac{X_{[n\cdot]}}{c\sqrt{n}\log n} \overset{J_1}{\Rightarrow} \max_{0\le s\le \cdot} B(s),$$

where $c > 0$ *is a constant.*

This statement might look puzzling: the process X_n, $n \ge 0$, is recurrent, yet the limiting process is transient. It is easier to believe that the above result holds if we replace $X_{[nt]}$ with its running maximum $X^*_{[nt]}$. The stated result comes from the fact that with an overwhelming probability the maximum amount of "backtracking" of X_j from X^*_j for $j \le [Tn]$ is of order $\sqrt{n}$, which is negligible on the scale $\sqrt{n}\log n$.

[16] Dolgopyat, Kosygina (2012) , arxiv)

Further results and open questions for $d = 1$

Assume (IID), (WEL) and (BD):

- Large deviations (under P_0) (Peterson (2012, arxiv)).
- The behavior of the maximum local time (Rastegar, Roiterstein, (2012, arxiv)).

Open question: Characterize the limiting behavior of ERW under the quenched measure $P_{0,\omega}$.