Potts and independent set models on d-regular graphs

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Outline

1. The Potts and independent set models
2. Locally tree-like graphs and the Bethe prediction
3. Previous work and results
4. Verifying the Bethe prediction: proof ideas

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Graphical models

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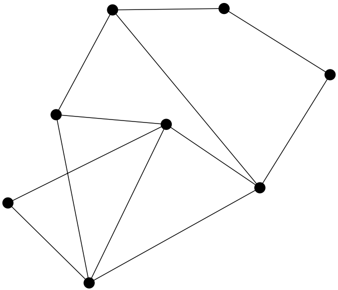
$G = (V, E)$ finite undirected graph



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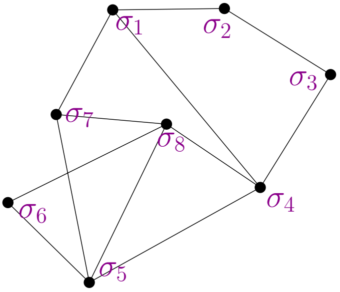

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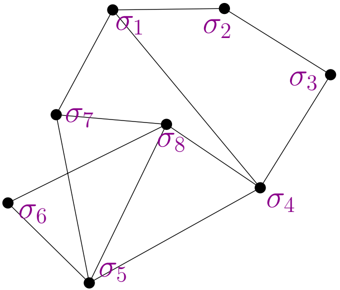

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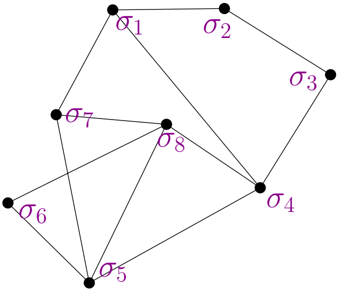

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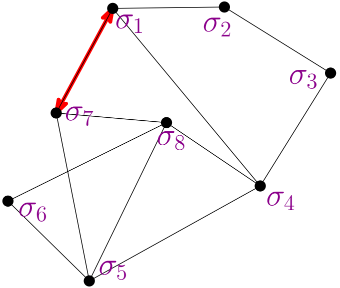
Model of random spin configuration defined by **local** interactions



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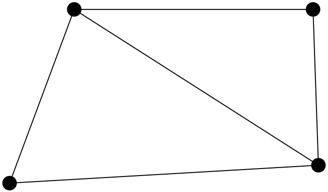

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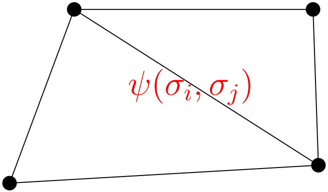

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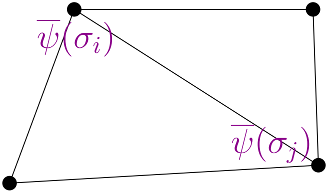

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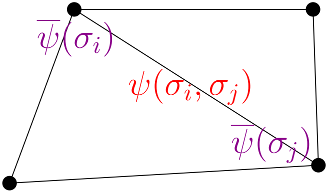

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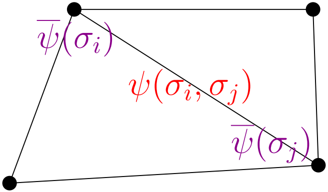

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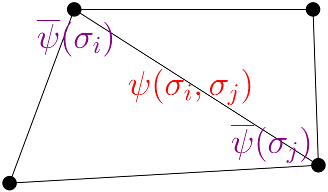

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Z = normalizing constant or **partition funtion**

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- $Z_G(\beta, B)$ = partition function

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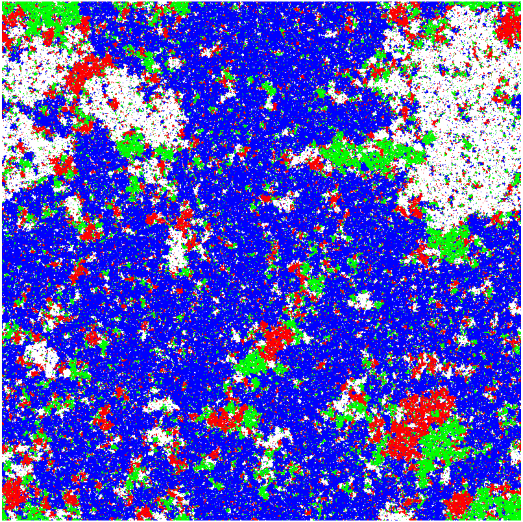

Figure: David Wilson

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- $\beta = -\infty$: random proper q-colorings

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  with $Z_G(1) =$ number of independent sets

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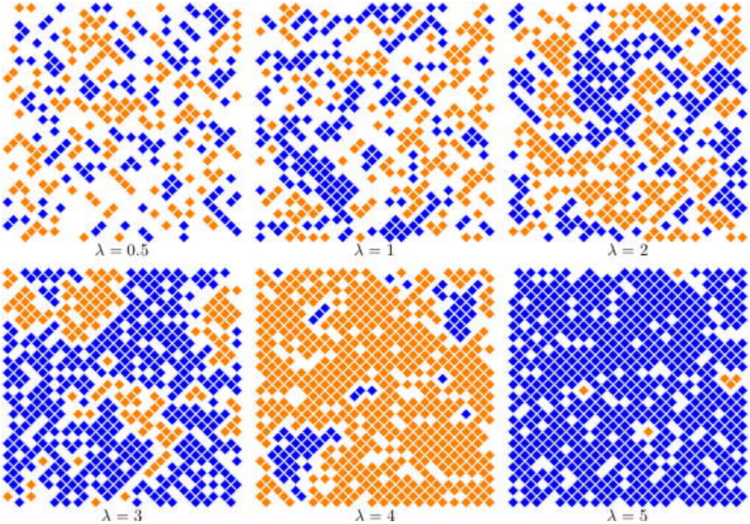

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The purpose of this work is to give an answer
in the setting of locally tree-like graphs

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Locally tree-like graphs

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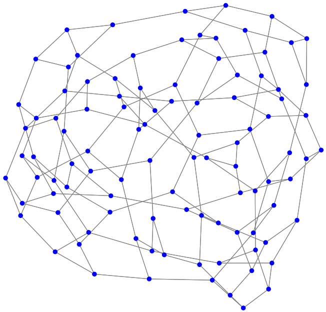
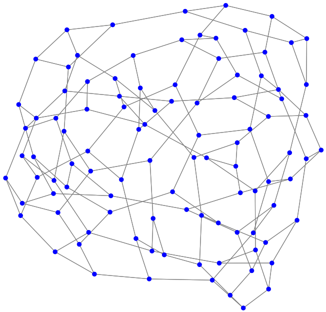


random 3-regular graph

In what sense is the random 3-regular graph locally like T_3?



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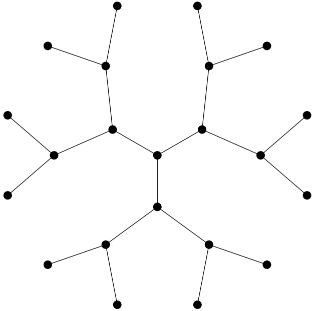

first few levels of T_3

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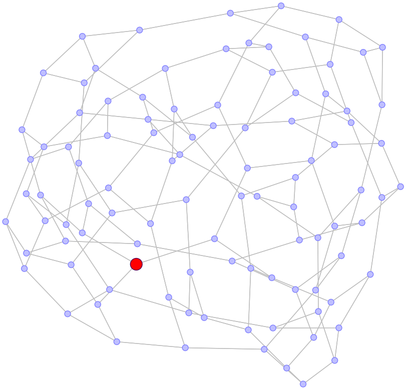

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Isomorphic to T_d^t
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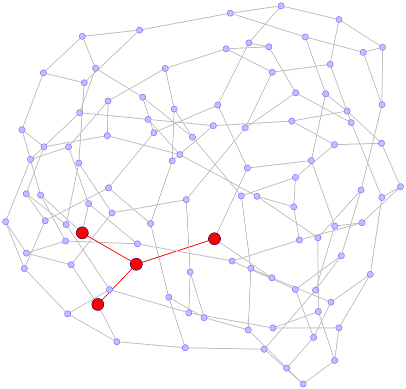

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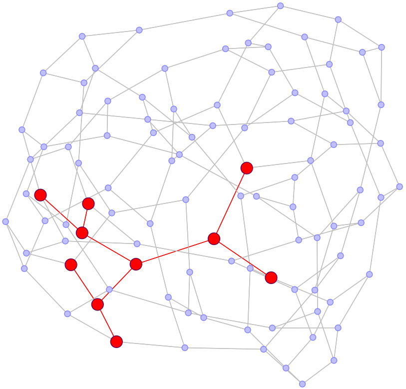

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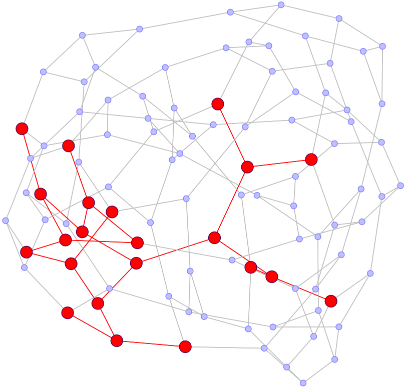

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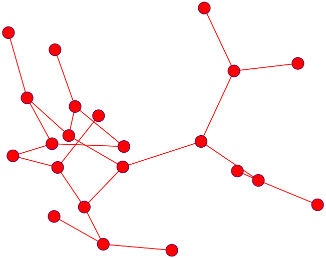

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[Can also make definition
with general (random) limiting tree]

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Local weak limits are unimodular measures
on the space of rooted graphs.

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Bethe prediction is defined only in terms of limiting tree — not the finite graphs G_n

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$$h(\sigma) \cong \bar{\psi}(\sigma) \left(\sum_{\sigma'} \psi(\sigma, \sigma') h(\sigma') \right)^{d-1}$$

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Functional form: $\Phi \equiv \Phi^{\mathsf{vx}} - \Phi^{\mathsf{e}}$ where

$$\Phi^{\mathsf{vx}} \equiv \log \left\{ \sum_{\sigma} \bar{\psi}(\sigma) \left(\sum_{\sigma'} \psi(\sigma, \sigma') h(\sigma') \right)^{d} \right\}$$

$$\Phi^{\mathsf{e}} \equiv \frac{d}{2} \log \left\{ \sum_{\sigma, \sigma'} \psi(\sigma, \sigma') h(\sigma) h(\sigma') \right\}$$

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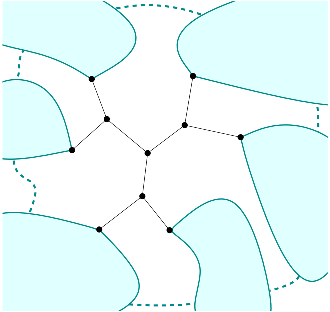

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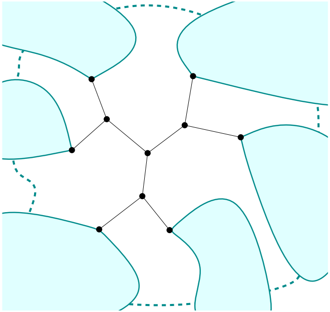

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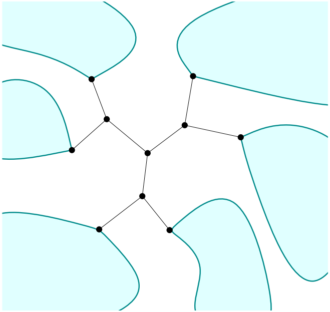

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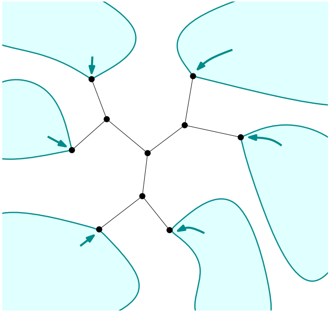

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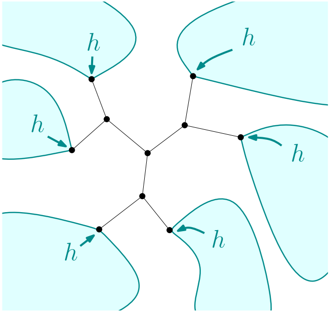

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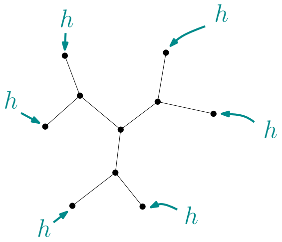

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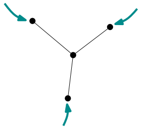

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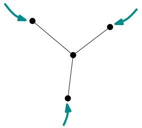

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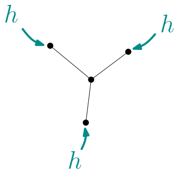

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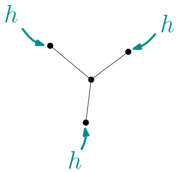

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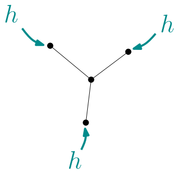

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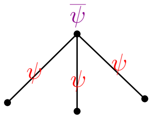

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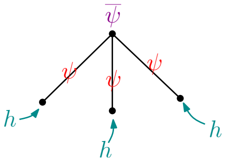

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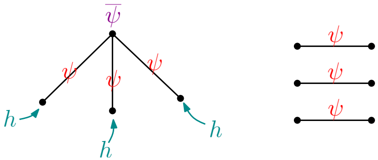

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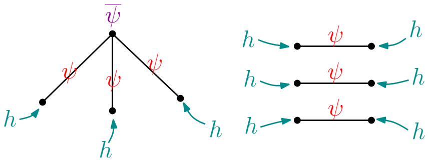

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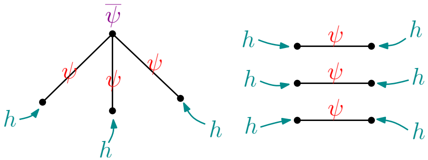

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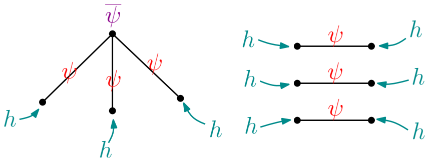

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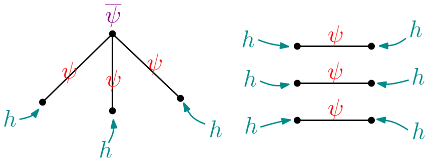

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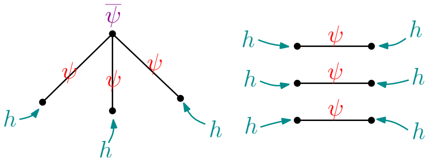

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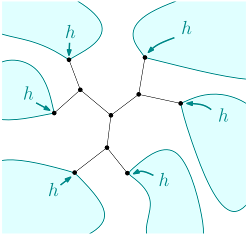

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For G_n non-bipartite, same prediction believed to hold
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2. Locally tree-like graphs and the Bethe prediction
3. Previous work and results
4. Verifying the Bethe prediction: proof ideas

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- FPTAS for AF Ising partition function $Z_G(\beta, B)$ on bdd. deg. graphs, $\beta_c^{\mathrm{af}}(B,d) < \beta < 0$ [Sinclair–Srivastava–Thurley '11]

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  [Galanis–Ge–Štefankovič–Vigoda–Yang '11]

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In independent work, Galanis–Štefankovič–Vigoda '12 established (a), and (b) with $B = 0$

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Can sometimes obtain ★ beyond uniqueness from
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By explicitly analyzing this mapping,
can obtain more exact results for T_d than are implied by interpolation scheme for general trees

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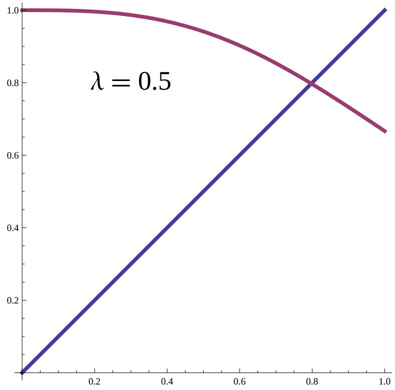
But by taking observable $i \mapsto (\sigma_i + d^{-1} \sum_{j \in \partial i} \sigma_j)/2$
can show $\phi = \Phi$ for all $\lambda > 0$

IS BP recursion

IS BP recursion (in terms of $h(0)$)



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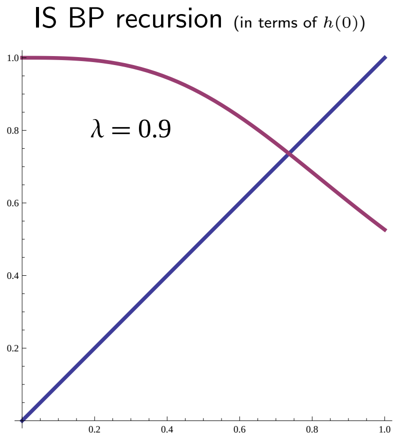

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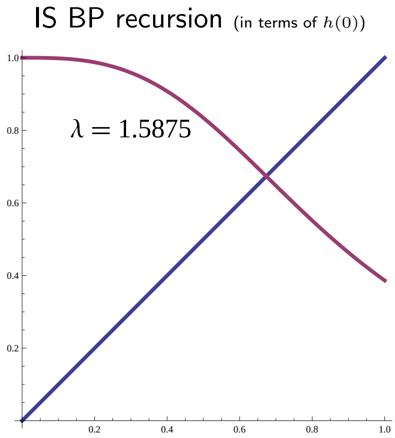

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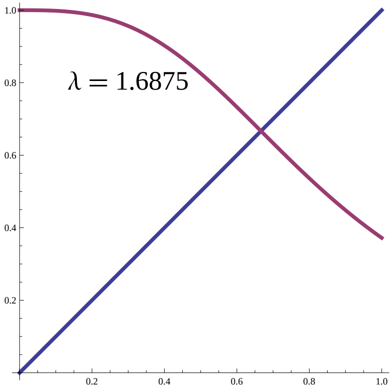

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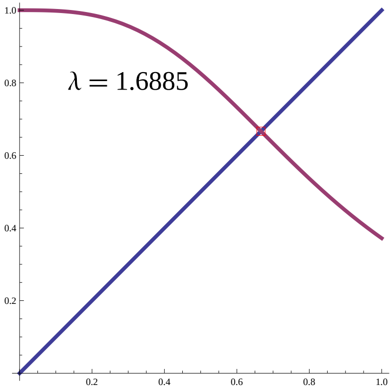

Semi-translation-invariant solutions arise above λ_c

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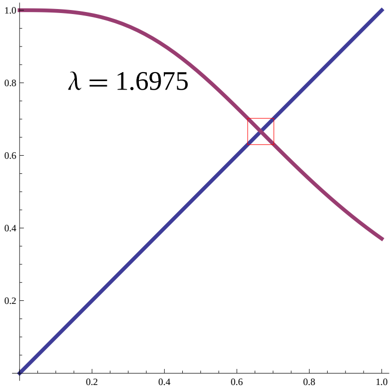

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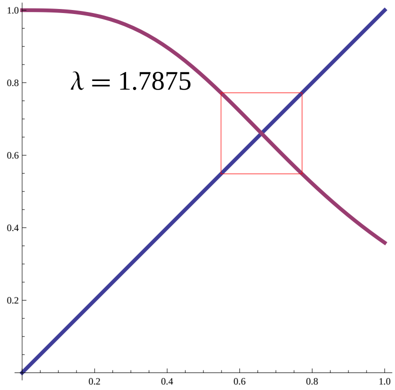

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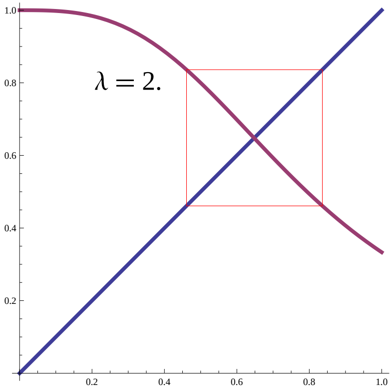

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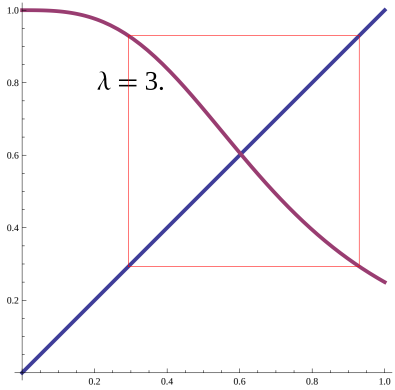

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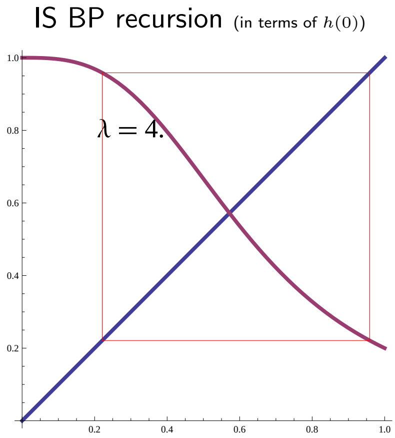

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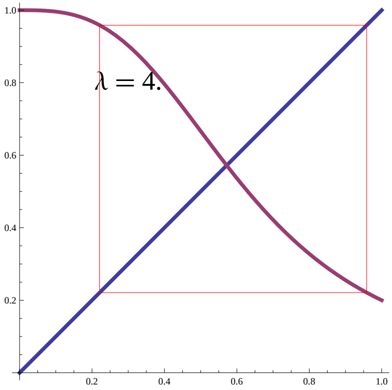
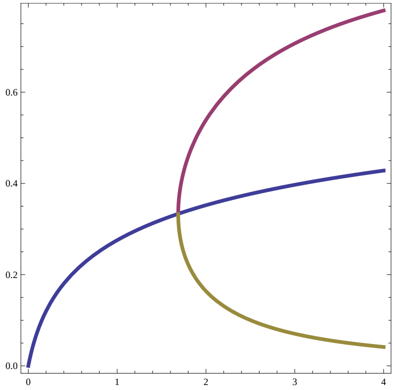


BP solutions as function of λ



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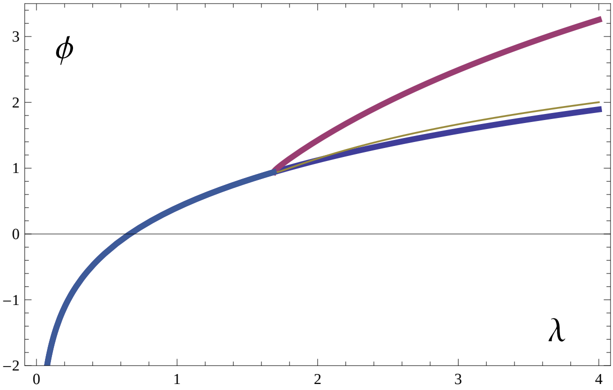

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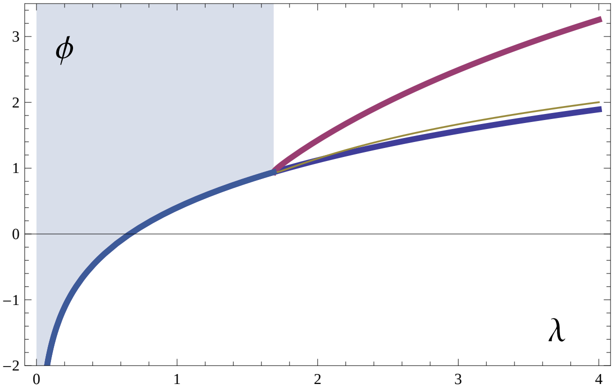

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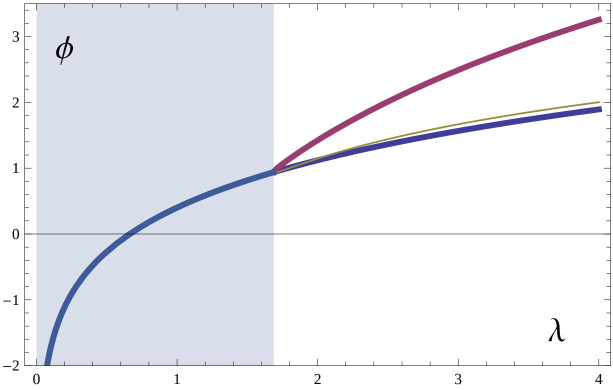

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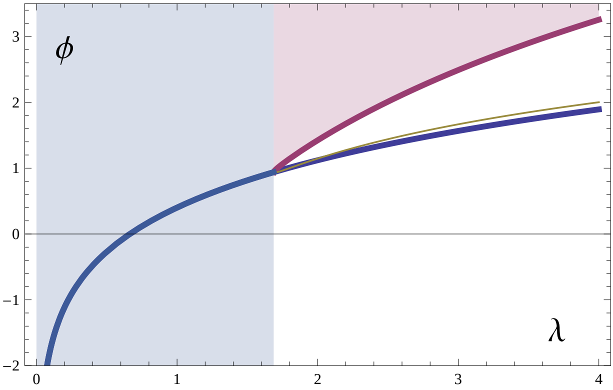

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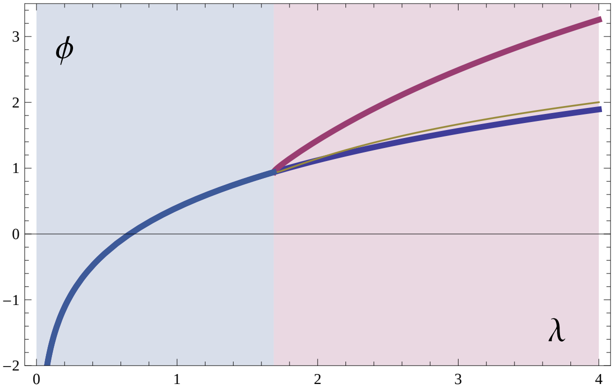

Use bipartite property to interpolate
semi-trans.-inv. fixed point from $\lambda = \infty$

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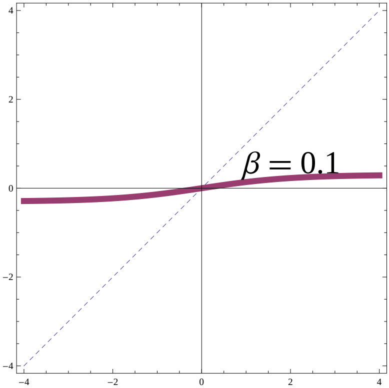
In Potts model, $\partial_B \phi_n = \mathbb{E}_n \mathbb{E}_{\nu_n}[\delta_{\sigma_{I_n},1}]$,
so local observable is simply $v \mapsto \delta_{\sigma_v,1}$

Similarly $\partial_\beta \phi_n = \mathbb{E}_n \mathbb{E}_{\nu_n}[\sum_{j \in \partial I_n} \delta_{\sigma_{I_n},\sigma_j}]$

In non-uniqueness regimes, can take advantage of **random-cluster** (FK) **representation** for Potts model to get monotonicity properties, thereby restricting range of admissible Gibbs measures

Ising vs. Potts

Ising BP (in terms of $\log[h(+)/h(-)]$)



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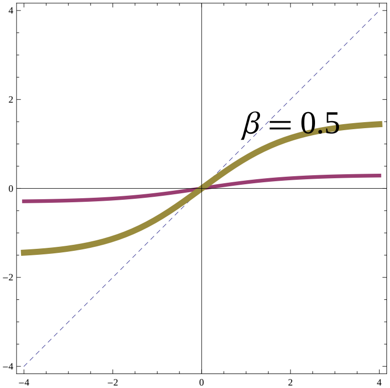

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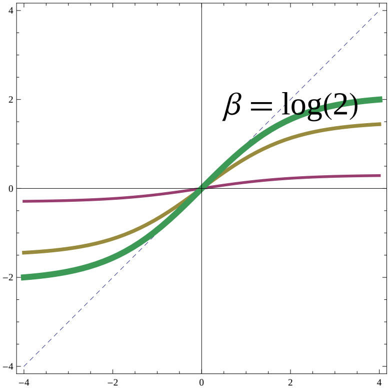

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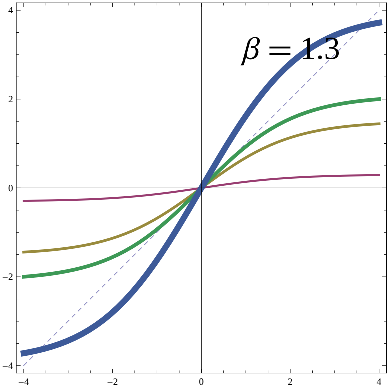

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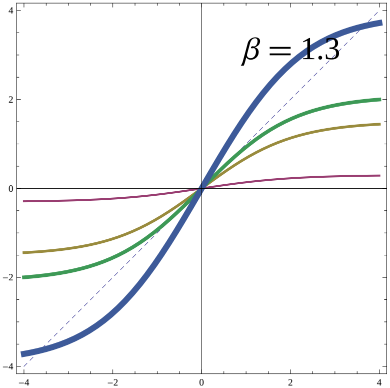
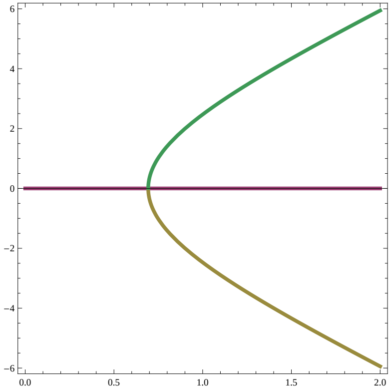


BP solutions as function of β



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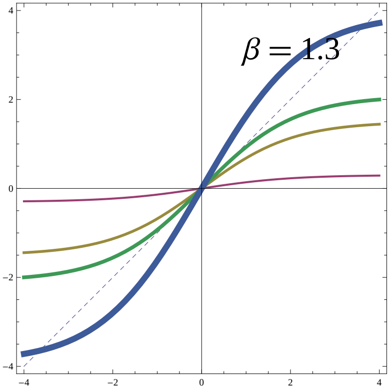

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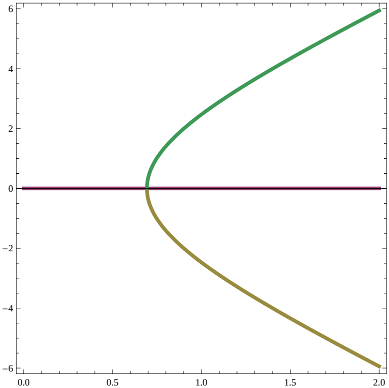

Adding small field $B > 0$ resolves non-uniqueness

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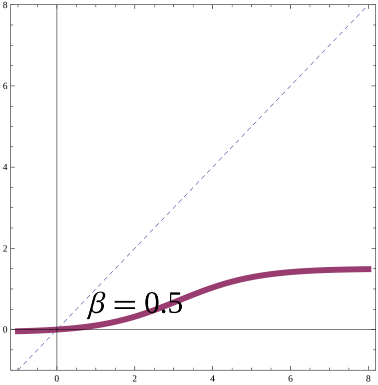

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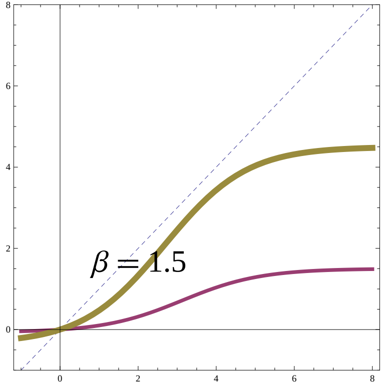

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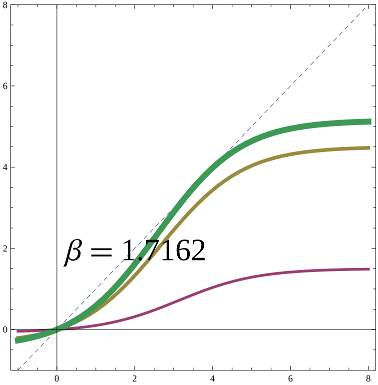

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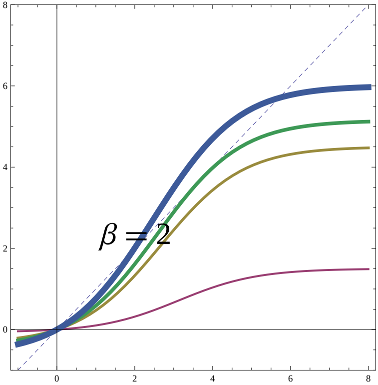

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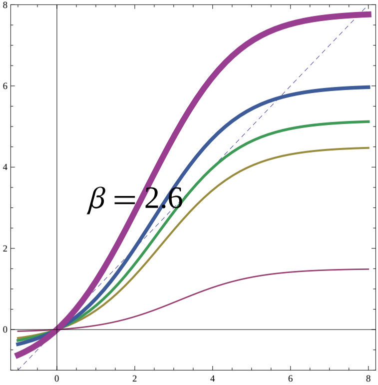

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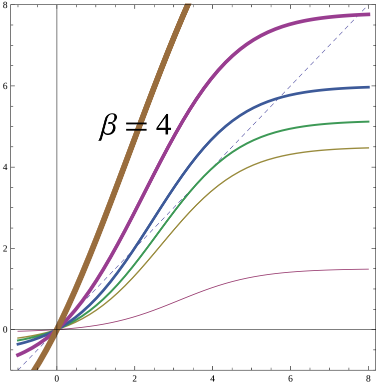

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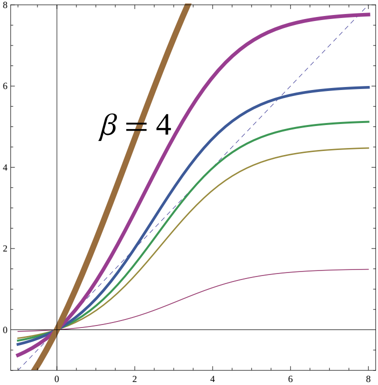

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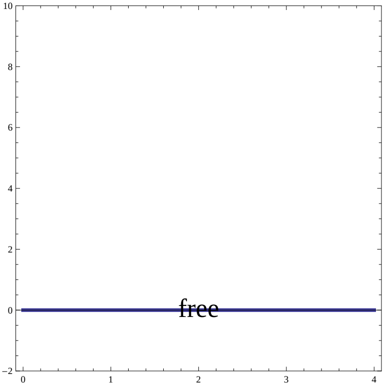

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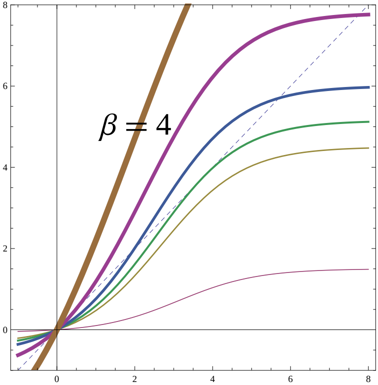

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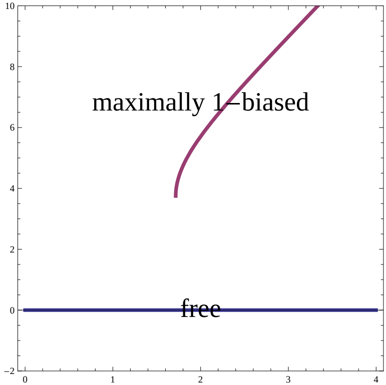

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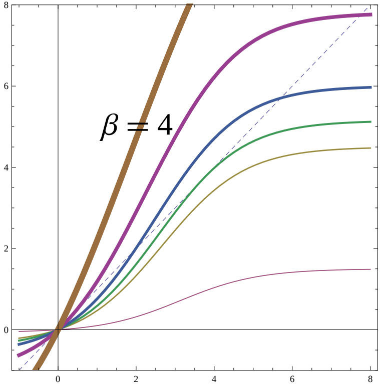

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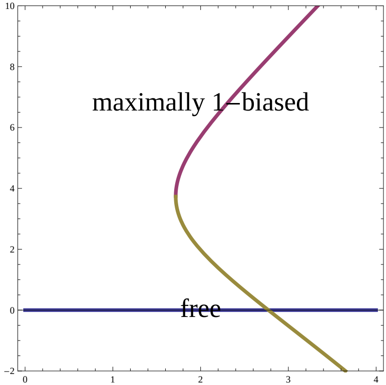

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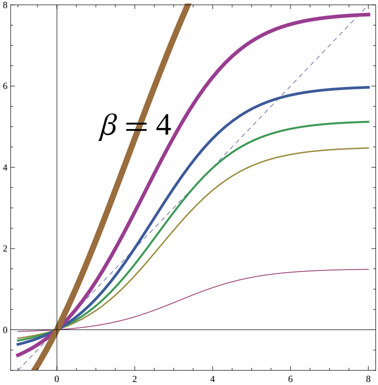

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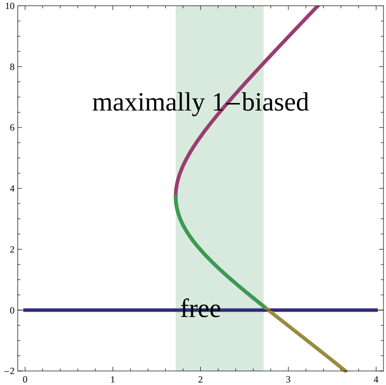

Adding $B > 0$ **not** enough to resolve non-uniqueness

Potts: $\phi \geq \Phi$ by interpolation

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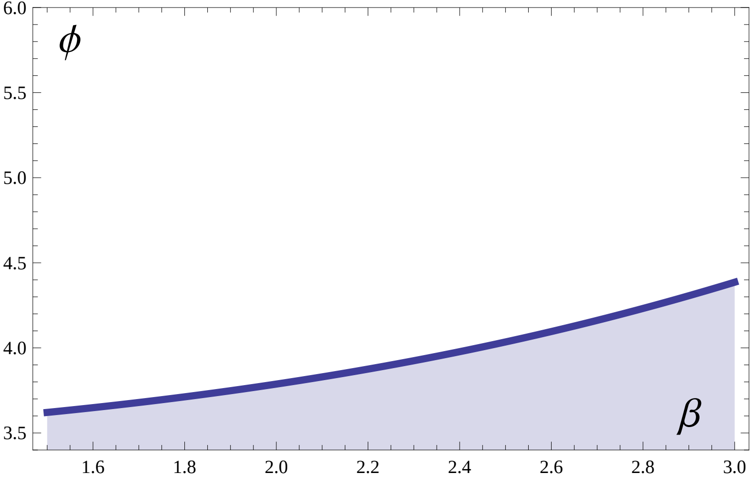

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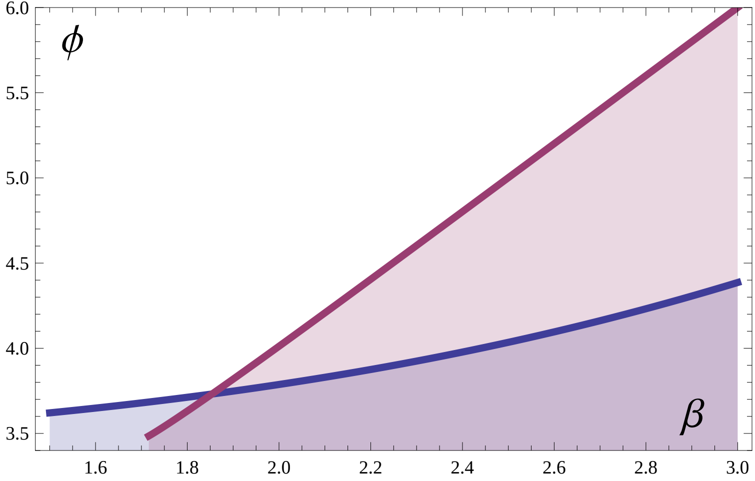

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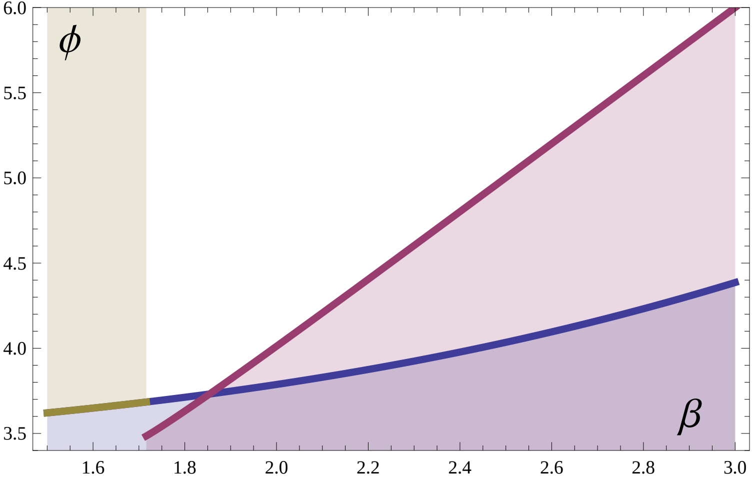

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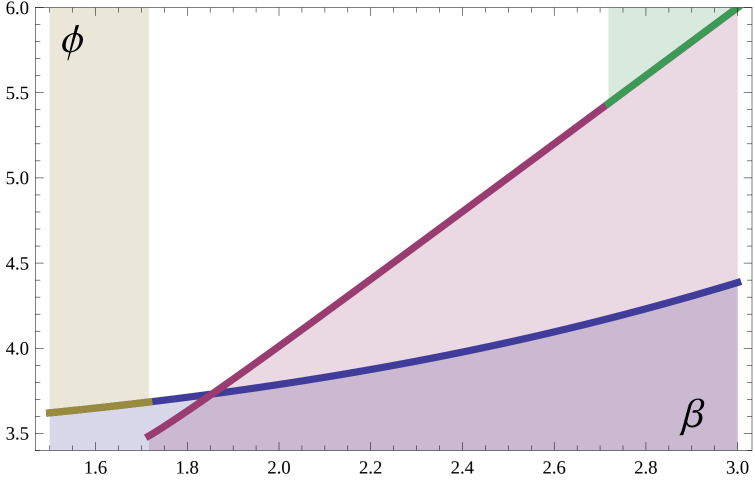

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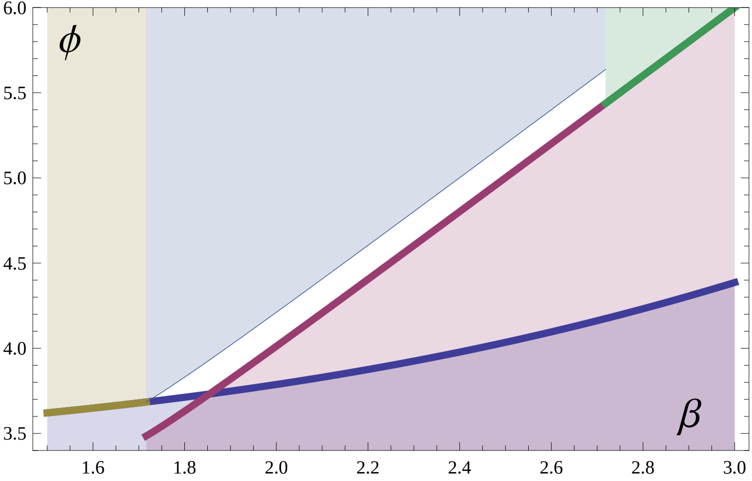

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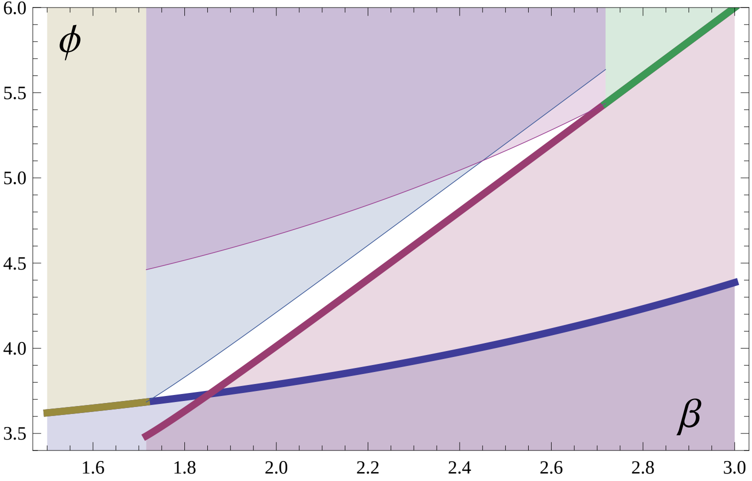

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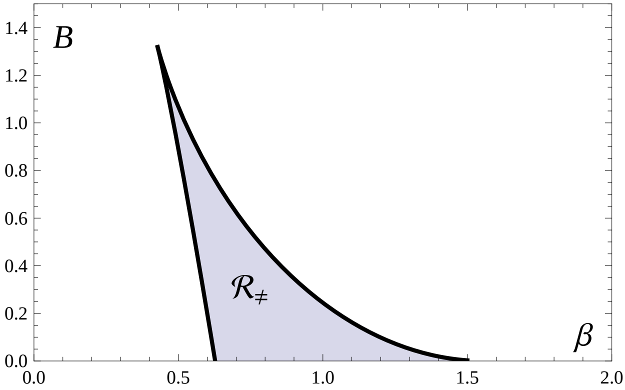

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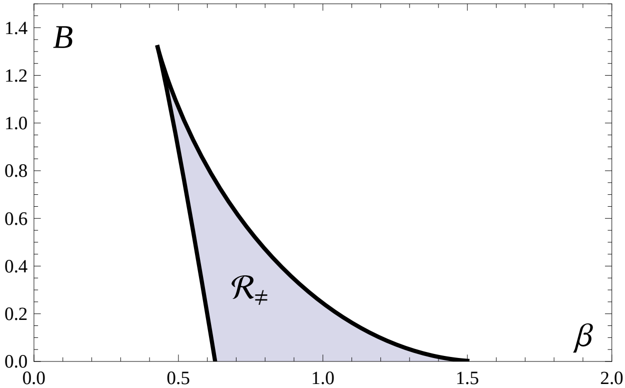

Different approach needed to obtain equality inside $\mathcal{R}_{\neq}$

Potts: $\phi \le \Phi$ by graph deconstruction

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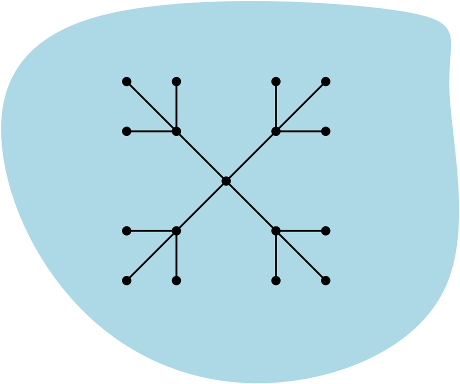

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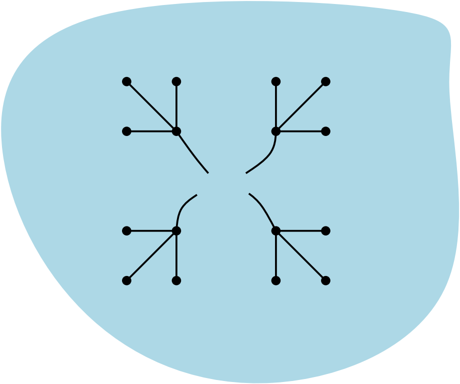

Delete a vertex

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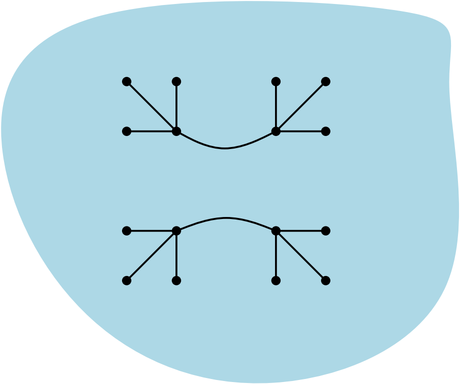

Delete a vertex

Match up half edges

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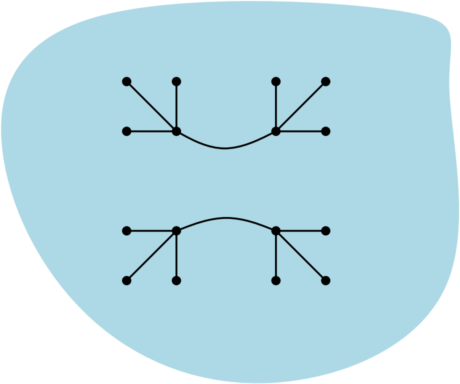
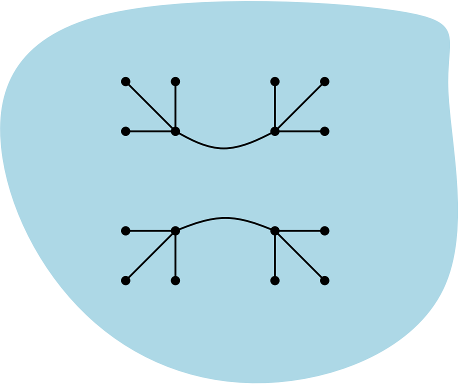


Delete a vertex

Match up half edges

Show decrease in $\log Z$ at each step is $\leq \Phi$ ★



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Matching **not** done u.a.r. but to guarantee ★

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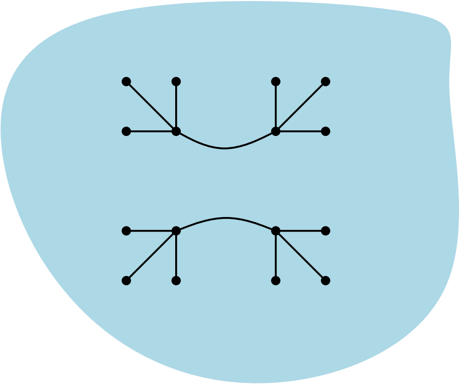

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Argue graphs remain uniformly locally tree-like

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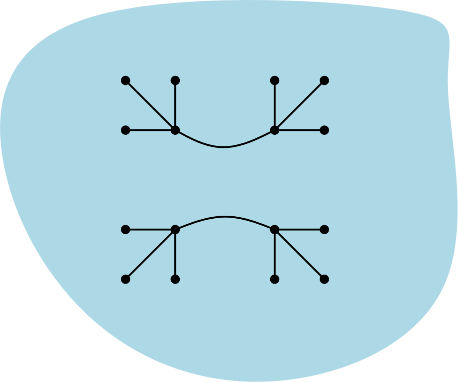

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Argue graphs remain uniformly locally tree-like

This procedure reduces the upper bound to showing ★, which is a difficult (but tractable) calculus problem

Two questions

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- The Bethe prediction is believed to be false for IS at high fugacity on typical non-bipartite graphs converging to T_d. Can one describe what happens in this case?

Thank you!