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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: BRANDEN STONE **Email/Phone:** bstone@bard.edu

Speaker's Name: ALEXANDER POLISHCHUK

Talk Title: LEFSCHETZ THEOREMS FOR dg-CATEGORIES W/ APPLICATIONS TO MATRIX FACTORIZATIONS

Date: 2/12/13 **Time:** 11:00 (am)/ pm (circle one)

List 6-12 key words for the talk: LEFSCHETZ, DG-CATEGORIES, MATRIX FACTORIZATIONS, COMPLETE INTERSECTIONS, HYPERSURFACE, ISOLATED SINGULARITY.

Please summarize the lecture in 5 or fewer sentances:

A description of versions of Lefschetz type formulas in the context of dg-categories. An explicit computation of the ingredients of such formulas in the case of the dg-category of matrix factorizations of an isolated hypersurface singularity is shown.

CHECK LIST

(This is **NOT** optional, we **will not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

LEFSCHETZ THEOREMS FOR dg-CATEGORIES WITH APPLICATIONS TO MATRIX FACTORIZATIONS

A. POLISHCHUK

TOPOLOGICAL LF: (M COMPLEX MANIFOLD)

$$f: M \longrightarrow M$$

$$str(f_*, H^*(X,\mathbb{Q})) := \sum (-1)^i tr(f_*, H^i(X,\mathbb{Q})) = \text{"\# Fix}(f)\text{"} = [\Delta]\cdot[\Gamma_f]$$

$\Gamma_f \cap \Delta$ TRANSVERSAL

(ATIYAH-BOTT) (SGA5) HOMOLOGICAL LF:

$$f: X \longrightarrow X,\ \text{HOMOMORPHISM},\ X \text{ IS COMPLEX MANIFOLD.}$$

$$\alpha: f^*V \longrightarrow V$$

$$str(\alpha, H^*(X,V)) := \sum_{x \in Fix(f)} tr\left(\alpha|_x, V|_x\right) \frac{1}{\det(1-d_x f)}$$

$$f = id,\quad \chi(X,V) = \int_X ch(V)\cdot Td_X$$

HRR FORMULA

MARKARIAN, CALDARARU, SHKLYAROV, ...

HRR FOR dg CATEGORIES

LUNTS: TLF FOR dg-CAT

THIS TALK: HLF — " —

dg-category:

$\mathrm{Hom}(\cdot,\cdot) \circlearrowleft d,\ d^2=1,\ \deg d = 1$

$D^b(\mathrm{Coh}\, X) \overset{\text{ENHANCEMENT}}{\rightsquigarrow}$ dg-category (Bondal–Kapranov)

A – dg-algebra / k – field

A – compact if $\dim H^*(A) < \infty$

$A^e = A^{op} \otimes_k A$, $\Delta_A = A$ viewed as a $A^{op}\otimes A$-dg-mod

$D(A)$ – derived category of dg-mod over A

$\cup$

$\mathrm{Per}(A)$ – smallest thick subcategory, containing A

A smooth: $\Delta_A \in \mathrm{Per}(A^e)$

$HH_*(A) = A \overset{\mathbb{L}}{\otimes}_{A^e} A = \mathrm{Tr}(\Delta_A)$

$\mathrm{Tr} : \mathrm{Per}(A^{op}\otimes A) \xrightarrow{\otimes^{\mathbb{L}}_{A^e} A} \mathrm{Per}(k)$

$\mathrm{Tr}(E^\vee \boxtimes F) = \mathrm{Hom}_A(E,F)$

Smooth compact

$(dg - Alg)$ - Bi category

$$Per(A^{op} \otimes B) \ni K \rightsquigarrow Per(A) \xrightarrow{K \otimes_A -} Per(B)$$

$$\Delta_A \in Per(A^{op} \otimes A) \rightsquigarrow Id_{Per(A)}$$

$$Tr_B(F \circ G) \simeq Tr_A(G \circ F)$$

$$A \underset{G}{\overset{F}{\rightleftarrows}} B$$

Functoriality of $HH_*(A)$

$$F: A \to B \Longrightarrow HH_*(A) \to HH_*(B)$$

$$HH_*(A) = Tr_A(\Delta_A) \to Tr_A(G \circ F)$$

$$\text{IS}$$

(F, G) - adjoint $\qquad Tr_B(F \circ G) \to Tr_B(\Delta_B)$

$$\| \qquad\qquad \| \qquad\qquad \| \qquad\qquad \|$$

$$HH_*(B) \qquad\qquad HH_*(B)$$

$E \in Per(A)$

$$k \xrightarrow{E} A \rightsquigarrow HH_*(k) \to HH_*(A)$$

$$\| \qquad\qquad \|$$

$$k: 1 \mapsto ch(E)$$

Canonical pairing

$$\langle , \rangle : HH_*(A^{op}) \otimes HH_*(A) \to k$$

$$Tr_{dg} : Per_{dg}(A^{op} \otimes A) \xrightarrow[\otimes_{A^e} A]{} Per_{dg}(k)$$

$$HH_*(A^{op} \otimes A) = HH_*(A^{op}) \otimes HH_*(A) \to k$$

HHR FOR dg-ALGEBRAS

$E, F \in \mathrm{Per}(A)$

$$\chi\left(H^{\bullet}\mathrm{Hom}(E,F)\right) = \left\langle \underset{\widehat{HH_*(A^{op})}}{ch(E^{\vee})}, \underset{\widehat{HH_*(A)}}{ch(F)} \right\rangle$$

$\underline{ch}$ $\quad\quad$ $\langle\ ,\ \rangle$

(A, F)

$f \in \mathrm{Per}(A^{op} \otimes A)$

$(A_1, F_1) \xrightarrow{\varphi} (A_2, F_2)$

$\varphi \in \mathrm{Per}(A_1^{op} \otimes A_2)$ & $\varphi \circ F_1 \longrightarrow F_2 \circ \varphi$

Replace $HH_*(A) = Tr(\Delta_A)$ with $Tr(F)$

(φ, ψ) - ADJOINT PAIR

$$\varphi_* : Tr_{A_1}(F_1) \longrightarrow Tr_{A_2}(F_2)$$

$$Tr_{A_1}(\psi\varphi F_1) \simeq Tr(\varphi F_1 \psi) \longrightarrow Tr(F_2 \varphi\psi)$$

COMPATIBLE WITH COMPOSITIONS!

A, F

$M \in \mathrm{Per}(A)$, $N \in \mathrm{Per}(A)$

$\alpha : M \longrightarrow F(M)$, $\beta : F(N) \longrightarrow N$

$$(F, \alpha, \beta)_* : \mathrm{Hom}(M,N) \xrightarrow{F} \mathrm{Hom}(F(M), F(N)) \longrightarrow \mathrm{Hom}(M,N)$$

$$str(F,\alpha\beta)_* = \langle \underset{\uparrow}{\tau(\alpha)}, \underset{\uparrow}{\tau(\beta)} \rangle_{F,G}$$

$$Tr(F) \qquad Tr(G)$$

(F,G) - ADJOINT

$$\langle \cdot,\cdot \rangle_{F,G} : Tr(F) \otimes Tr(G) \longrightarrow k$$

$$A \sim D^b(Coh\, X)$$

$F = f^*$, $f: X \to X$

$f^* Y \to Y$ $\qquad \Delta \cap \Gamma_f$ - TRANSVERSAL

$$Tr(f_*) = Tr(f^*) = \bigoplus_{x \in Fix} k \cdot \delta_x$$

$$\langle \delta_x, \delta_x \rangle = \frac{1}{\det(1 - d_x f)}$$

$w \in R$ - COMMUTATIVE

$$MF(w): \; P_0 \underset{\delta}{\overset{\delta}{\rightleftarrows}} P_1, \quad \delta^2 = w \cdot id_P$$

$\bar{P} = (P, \delta)$ $\qquad Hom(\bar{P}, \bar{Q}) = Hom_R(P, Q)$

d (graded), $d^2 = 0$ $\qquad [\delta, -]$

$\mathbb{Z}/2$ - GRADED dg-CATEGORY

$R = k[[x_1, \dots, x_n]]$, w = ISOLATED SING $\Longrightarrow$ $MF(w)$ IS SMOOTH AND COMPACT

$t = (t_1, \dots, t_n)$, $w(t_1 x_1, \dots, t_n x_n) = w(x_1, \dots, x_n)$

TRANSVERSAL $t_i \neq 1$

$t^*: MF(W) \circlearrowleft$

$\alpha: \overline{E} \xrightarrow{\sim} t^*\overline{E}$, $\beta: t^*\overline{E} \to \overline{E}$

$$str\Big((\alpha,\beta): t^*|_{MF(W)}\Big(\overline{E},\overline{F}\Big)\Big) = str\Big(\alpha|_0, E|_0\Big)\cdot str\Big(\beta|_0, F|_0\Big) \cdot \prod_{i=1}^{n}(1-t_i)^{-1}$$

Ex: $t=-1$

$W(-x) = W(x)$

$\overline{E}$ - $\mathbb{Z}/2$ - equiv. m.f. of W

$$\Longrightarrow \quad Z^{\left[\frac{n}{2}\right]}\Big| \; str\Big((-1)^*, E|_0\Big)$$