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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: BRANDEN STONE Email/Phone: bstone@bard.edu

Speaker's Name: TOBIAS DYCKERHOFF

Talk Title: HIGHER SEGAL SPACES

Date: 2/14/13 Time: 11:30 am / pm (circle one)

List 6-12 key words for the talk: SEGAL SPACES, HALL ALGEBRA, REPRESENTATION THEORY, HECKE ALGEBRA,

Please summarize the lecture in 5 or fewer sentances:
An outline of some aspects of the theory of higher Segal spaces.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

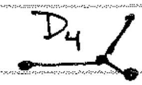
Higher Segal Spaces

T. Dyckerhoff

(1) Hall Algebras

$\mathcal{A}$ abelian category

- $|\mathrm{Hom}(A, A')| < \infty$
- $|\mathrm{Ext}^1(A, A')| < \infty$


D_4

Ex: $\mathrm{Vect}_{\mathbb{F}_q}$, Ab_p, $\mathrm{Rep}_{\mathbb{F}_q}(Q)$, ADE-Dynkin

Def: $$\mathrm{Hall}(\mathcal{A}) := \bigoplus_{[A] \in \pi_0(\mathcal{A})} \mathbb{C} \cdot [A]$$

$$[A] \cdot [A'] := \sum g^{B}_{AA'} [B]$$

$$g^{B}_{AA'} := \frac{|\{0 \to A \to B \to A' \to 0\}|}{|\mathrm{Aut}(A)|\ |\mathrm{Aut}(B)|}$$

Fact: $(\mathrm{Hall}(\mathcal{A}), \cdot)$ is an associative algebra

Example: $\mathcal{A} = \mathrm{Vect}_{\mathbb{F}_q}$

$$\mathrm{Hall}(\mathrm{Vect}_{\mathbb{F}_q}) = \bigoplus_{n \geq 0} \mathbb{C} \cdot [\mathbb{F}_q^n]$$

$$[\mathbb{F}_q^n] \cdot [\mathbb{F}_q^m] = \left\{ \begin{matrix} n\text{-dim} \\ \text{subspaces} \\ \text{of } \mathbb{F}_q^{n+m} \end{matrix} \right\} \cdot [\mathbb{F}_q^{n+m}]$$

$$\| $$

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q$$

$$(n)_q := 1 + q + q^2 + \dots + q^{n-1}$$

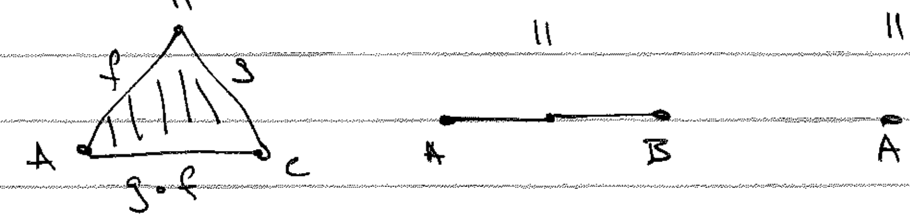
<u>HISTORY</u>: STEINITZ (1901), HALL (1959), RINGEL (1990), $\longrightarrow$

$\downarrow$ ~~HIGHER ALGEBRAIC STRUCTURES~~

<u>2-SEGAL SPACES</u>

$$\{0<1<2<\cdots<n\}$$
$$\|$$
$$C\text{ - SMALL CATEGORY} \rightsquigarrow N(C)_n := \mathrm{Fun}([n], C)$$

$$\cdots\ \substack{\rightarrow\\\rightarrow\\\rightarrow} N(C)_2 \ \substack{\rightarrow\\\rightarrow\\\rightarrow} N(C)_1 \rightrightarrows N(C)_0$$



$$(*) \qquad N(C)_n \xrightarrow{\cong} N(C)_1 \times N(C)_1 \times \cdots \times N(C)_1$$

$$\Delta^n \hookleftarrow$$

SEGAL CONDITION

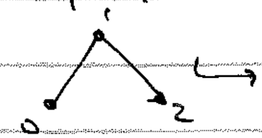
$$\{\text{SMALL CATEGORIES}\} \xrightarrow[\simeq]{N} \left\{\begin{matrix}\text{SIMPLICAL SETS}\\ \text{SATISFYING } (*)\end{matrix}\right\}$$
$$\|$$
$$\text{SEGAL SETS}$$

<u>REZK</u>: A SIMPLICIAL SPACE X IS CALLED SEGAL SPACE IF $\forall n$:

$$X_n \xrightarrow{\sim} X_1 \times X_1 \times \cdots \times X_1$$

ARE HOMOTOPY EQUIVALENCES.

$$X_1 \times X_1 \xleftarrow{\sim} X_2$$

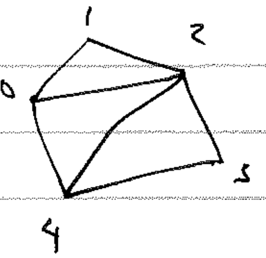


$$\hookrightarrow \Delta^2$$

$$\{\infty\text{ - CATEGORY}\} \longleftrightarrow \left\{\begin{matrix}\text{(COMPLETE)}\\ \text{SEGAL SPACES}\end{matrix}\right\}$$

JOYAL - TIERNEY

Consider

$\Delta^4 \hookleftarrow$  $=: \tau$

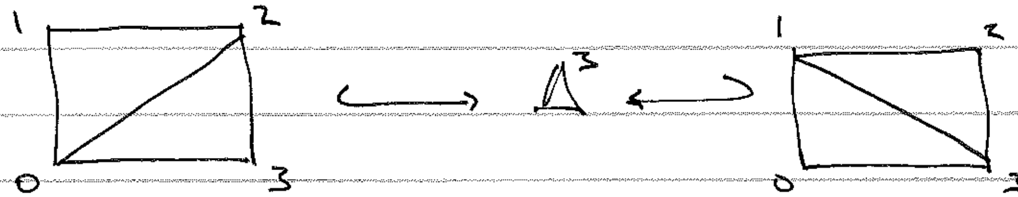
$$X_4 \longrightarrow X_2 \times_{X_1} X_2 \times_{X_1} X_2 =: X_\tau$$

Def: A simplicial space X is called 2-Segal
$\forall n \geq 2$: $\forall \tau$ triangles of all convex $(n+1)$-gons

$$X_n \xrightarrow{\sim} X_\tau$$

homotopy equivalence

Lowest 2-Segal conditions:



$$X_2 \times_{X_1} X_2 \xleftarrow{\simeq} X_3 \xrightarrow{=} X_2 \times_{X_1} X_2$$

$$m_2(m_2 \otimes 1) \longleftarrow\!\shortmid m_3 \longmapsto m_2(1 \otimes m_2)$$

Example: Every 1-Segal space is a 2-Segal space

$\mathcal{A}$ an abelian category: $S_\cdot(\mathcal{A})$

$$\left|\begin{bmatrix} 0 \hookrightarrow A \hookrightarrow B \\ \quad\ \downarrow \qquad\ \downarrow \\ \quad\ 0 \hookrightarrow A' \end{bmatrix}\right| \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \left|\begin{bmatrix} 0 \hookrightarrow A \\ \quad\ \downarrow \\ \quad\ 0 \end{bmatrix}\right| \rightrightarrows |[0]|$$

$$\uparrow \uparrow \uparrow \uparrow$$

$$\left|\begin{bmatrix} 0 \hookrightarrow A \hookrightarrow B \hookrightarrow C \\ \quad\ \downarrow \qquad \downarrow \qquad \downarrow \\ \quad\ 0 \hookrightarrow A' \hookrightarrow B' \\ \qquad\qquad \downarrow \qquad \downarrow \\ \qquad\qquad 0 \hookrightarrow A'' \end{bmatrix}\right|$$

Waldhausen: $\Omega |S_\cdot(\mathcal{A})| = K(\mathcal{A})$

THM: The Waldhausen S-construction of any exact ∞-category is a 2-Segal space

Cor: Associativity of classical, motivic, derived, Hall alg.

More Precisely: X 2-Segal space, h transfer theory and finite cond, $\mathrm{Hall}(X,h)$

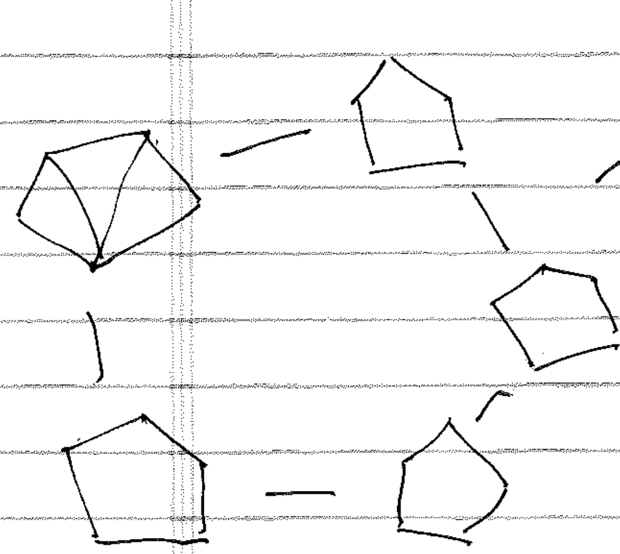
Other Examples

- Hecke-Waldhausen construction
  $\Longrightarrow$ Hecke Alg
- $N^{cyc}_{\infty\text{-cat}}(C)$ 2-Segal $\Longrightarrow$? Algebras

Thm: X 2-Segal space

- $\exists$ monoidal ∞-category of spans of spaces
- From X we can obtain A_X which is an A_∞-alg in $\mathrm{Span}(\text{spaces})^{\otimes}$

Applications of higher 2-Segal conditions



$$\begin{array}{c} \rightsquigarrow \left(\otimes, \mathrm{Fun}_0\left(\coprod_{[A]\in\mathcal{A}} B\mathrm{Aut}(A), \mathrm{Vect}_{\mathbb{C}}\right)\right) \\ \downarrow \\ (K_0(-), \cdot)\otimes\mathbb{C} \\ \downarrow{\scriptstyle \text{rank}} \\ \mathrm{Hall}(\mathcal{A}) \end{array}$$

Cyclic 2-Segal Spaces

Thm: X cyclic 2-Segal space, Σ is punctured, closed, oriented surface, then $\exists$ a space

$$X_\Sigma \circlearrowleft \quad \exists \text{ action of mapping classing group}$$

Example:

M monoid $\rightsquigarrow N^{cyc}(M)$ 2-Segal, cyclic

$$\Sigma = (\text{Forms}, *) \underset{SL(2,\mathbb{Z})}{\rightsquigarrow} \circlearrowleft X_\Sigma = \left\{ (a,b,c,d) \;\middle|\; \begin{matrix} dab = a \\ cbc = b \\ bcd = c \\ cda = d \end{matrix} \right\}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : (a,b,c,d) \longmapsto (d,a,b,c)$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : (a,b,c,d) \longmapsto (ab, b, cd, d)$$

Thm: The Waldhausen S-construction of a $\mathbb{Z}/2$-graded triangulated dg-cat. admits a canonical cyclic structure

$$\begin{matrix} \mathbb{Z}/4 \circlearrowleft & & \mathbb{Z}/3 \circlearrowleft & & \mathbb{Z}/2 \circlearrowleft & & \\ X_3 & \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} & X_2 & \substack{\rightarrow\\\rightarrow\\\rightarrow} & X_1 & \rightrightarrows & X_0 \end{matrix}$$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \arrow[d] & & \\
& 0 \arrow[r] & A' \arrow[r] \arrow[d] & A[1] \arrow[r] \arrow[d] & 0 \arrow[d] & \\
& & 0 \arrow[r] & B[1] \arrow[r] \arrow[d] & A'[1] \arrow[d] & \\
& & & 0 \arrow[r] & A \arrow[r] & B \arrow[r] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & 0 \arrow[d] & \\
& 0 \arrow[r] & A[1] \arrow[r] \arrow[d] & 0 \arrow[d] \\
& & 0 \arrow[r] & A[2] \simeq A
\end{tikzcd}$$