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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: BRANDEN STONE **Email/Phone:** bstone@bard.edu

Speaker's Name: ~~[illegible]~~ ~~[illegible]~~ HAILONG DAO

Talk Title: COHEN-MACAULAY CONES & SUBCATEGORIES

Date: 2 / 14 / 13 **Time:** 10:30 am / pm (circle one)

List 6-12 key words for the talk: COHEN-MACAULAY, CONES, SUBCATEGORIES, MAXIMAL COHEN-MACAULAY, REPRESENTATION TYPE, GROTHENDIECK GROUP

Please summarize the lecture in 5 or fewer sentances:

Results on maximal Cohen–Macaulay (MCM) modules over local CohenMacaulay rings coming from two different projects. The first studies different invariants of resolving subcategories of MCM modules. The second project is about the cone of MCM modules in the Grothendieck group modulo numerical equivalences.

CHECK LIST

(This is **NOT** optional, we will **not pay for incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
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- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Cohen-Macaulay cones and subcategories

Hailong Dao

University of Kansas

March 12, 2013

Foreword

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  1. Size of $CM(R)$ and singularities of $\operatorname{Spec} R$.
  2. Classification of resolving subcategories of $\operatorname{mod}(R)$ over complete intersections.
  3. NEF and CM cones in $G(R)$ modulo numerical equivalences.

1. Smaller $\mathrm{CM}(R)$ = better singularities

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(Auslander-Buchsbaum-Serre) R is regular if and only if $\mathrm{CM}(R) = \{freemodules\}$. In particular there is only **one** *indecomposable object in it.*

- So, we count (the indecomposable objects): Auslander, Buchweitz-Greuel-Schreyer, Knörrer on finite type. Many other people count, too.

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- New ways to measure size: inspired by definition of dimension of triangulated categories (Bondal-Van den Bergh, Rouquier).
- Ball of radius n: $[X]_n$. Dimension of a subcategory.

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Figure : Extensions = sushi pieces, direct summands = soy sauce, etc

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Not counting this way is like enjoying a buffet:

Figure : Happier times

Examples:

Proposition

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring.

(1) *If R has finite Cohen-Macaulay representation type then* $\dim \mathrm{CM}(R) = 0$. *The converse is true of R is hensenlian and Gorenstein.*

(2) *Suppose R is a complete local hypersurface with an algebraically closed coefficient field of characteristic not two. If R has countable Cohen-Macaulay representation type, then* $\dim \mathrm{CM}(R) = 1$.

(3) *Suppose* $\dim R = 2$, *k is algebraically closed of char 0 and R is hensenlian, normal with rational singularity. Then* $\dim \mathrm{CM}(R) \leq 1$. *It is 0 if and only if R has simple (quotient, Du Val, Kleinian,...) singularities.*

Proof.

(of 3) By Iyama-Wemyss, there exists $X \in \mathrm{CM}(R)$ such that $\Omega\mathrm{CM}(R) = \operatorname{add} X$. Let $M \in \mathrm{CM}(R)$ be any maximal Cohen-Macaulay module and let $M^\vee$ denote $\mathrm{Hom}_R(M, \omega_R)$. We have an exact sequence:

$$0 \to \Omega(M^\vee) \to R^n \to M^\vee \to 0$$

Applying $\mathrm{Hom}_R(-, \omega_R)$ we get

$$0 \to M \to \omega_R^n \to (\Omega(M^\vee))^\vee \to 0$$

It follows that $\mathrm{CM}(R) = [X^\vee \oplus \omega_R]_1$. $\square$

1. Smaller $\mathrm{CM}(R)$ = better singularities

Theorem

(D, Takahashi) Let R be a Cohen-Macaulay local ring and $V \subseteq \operatorname{Spec} R$ is closed. Consider the following conditions.

(a) $\dim \mathrm{CM}_V(R)$ *is finite.*

(b) $\mathrm{Sing}(R) \subseteq V$ *(equivalently,* $\mathrm{CM}_V(R) = \mathrm{CM}(R)$*).*

Then (a) $\Rightarrow$ (b). *The implication* (b) $\Rightarrow$ (a) *also holds if R is complete, equicharacteristic and with perfect residue field or essentially of finite type over any field.*

Corollary

Let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$. Consider the following conditions.

(a) $\dim \mathsf{CM}_{\mathfrak{m}}(R)$ *is finite.*

(b) *R has isolated singularity.*

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Corollary

If $\mathsf{CM}_{\mathfrak{m}}(R)$ has finite type then R has isolated singularity.

Questions/Remarks

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- What about $\dim \mathrm{CM}(R)$ for Gorenstein quotient singularities? It is known to be 1 in dimension 3 and bounded above by $d-2$ in higher dimensions. For other singularities? For cones over curves of higher genus?

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Theorem

(the Village) Let $R = S/(\mathbf{x})$ where S is a regular ring and $\mathbf{x} = x_1, \dots, x_c$ is a regular sequence on S. Set $Y = \operatorname{Proj} S[y_1, \cdots, y_c]/(\sum x_i y_i)$. Then one has a 1-1 correspondence

$$\left\{\begin{matrix} \text{Resolving subcategories} \\ \text{of} \quad \operatorname{mod} R \end{matrix}\right\} \begin{matrix} \longrightarrow \\ \longleftarrow \end{matrix} \left\{\begin{matrix} \text{Specialization closed} \\ \text{subsets of } \operatorname{Sing} Y \end{matrix}\right\} \times \left\{\begin{matrix} \text{Grade consistent} \\ \text{functions on } \operatorname{Spec} R \end{matrix}\right\}.$$

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- Example: M, N finite length modules. Then N can be built from M using direct summands, extensions, and syzygies if and only if the support variety of N sits inside that of M.
- Can one estimate how many steps we need using computable invariants?
- What about other rings? Conjecture: can't be classified just with geometric objects in general.

2. CM(R) is resolving

3. Cohen-Macaulay cones

Figure : How the story started

3. Cohen-Macaulay cones

Let X be a projective variety over a field. A *cycle* is a formal sum of subvarieties of X. The Chow group of X, $\mathrm{CH}(X)$ is the free abelian group of all cycles modulo *rational equivalences*. When X is smooth, it turns out that we can turn it into a commutative ring with a suitable intersection product. A cycle is said to be *numerically trivial* if its product with all cycle of the complimentary dimension is zero. By $\overline{\mathrm{CH}}(X)$ we denote the Chow group of X modulo the subgroup generated by numerically trivial elements. Important fact : $\dim_{\mathbb{Q}} \overline{\mathrm{CH}}(X)_{\mathbb{Q}}$ is *finite*. It is an Artinian $\mathbb{Q}$-algebra!

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- Switch to Grothendieck groups.
- Use Serre's intersection multiplicities.

Generalized Serre's intersection multiplicity

Let $(R, \mathfrak{m})$ be a (comutative noetherian) local ring and M, N be R-modules. All modules are finitely generated.

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Definition

(Serre, 1961) Assume that $\operatorname{pd} M < \infty$ *and* $M \otimes N$ *has finite length. One defines:*

$$\chi^R(M, N) = \sum_{i \geq 0} (-1)^i \ell(\operatorname{Tor}_i^R(M, N))$$

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When R is regular and contains a field, Serre proved that this definition gives what one expects from intersection theory (vanishing, positivity, etc..).

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- Slight problem: the cone $C(R)$ is hard to compute.
- Example: $C(R)$ for $R = \mathbb{C}[[x, y, u, v]]/(xy - uv)$, due to Dutta-Hochster-McLaughlin and Knörrer.

3. Cohen-Macaulay cones

Theorem

Let R be a local hypersurface with isolated singularity and $\dim R = 3$*. Assume that* $\operatorname{Spec} R$ *admits a resolution of singularity.*

1. $\overline{\mathrm{G}}(R) = \mathbb{Z}[R] \oplus A_2(R) \cong \mathbb{Z}[R] \oplus \mathrm{Cl}(R)$.
2. *If* $R = k[[x, y, u, v]]/(xy - f(u, v))$ *(*cA_n *singularities) then* $C(R)$ *is generated by the all the rays* $\mathbb{R}_{\geq 0}[(x, g)]$ *where* g *is a irreducible factor of* f*.*

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Proof: sufficient criteria for numerical functions on $\mathrm{G}(R)$ that arise from perfect complexes with finite length homologies, positive-definiteness of Hochster's theta invariant, Knörrer's periodicity.

Remarks

Let's revisit the first example. Let $R = k[x, y, u, v]/(xu - yv)$. What is the "singularity" of R?

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- Non-commutative: $\mathrm{Hom}_R(R \oplus I)$ has finite global dimension.
- Commutative algebra: $M = R/I$ serves as a counter-example to many statements of the homological conjectures (if the ring is not regular and we don't make extra assumptions).

Thus, various different points of view suggests that the true singularity of R is actually "concentrated" in (the class of) I! Similar pictures exist for $R = k[[x, y, u, v]]/(xy - f(u, v))$, due to the work of many.

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Let R be a Cohen-Macaulay domain. The boundary of the closure of $C(R)$ has no intersection with the hyperplane H generated by the torsion modules except at the origin.

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Corollary

The set of projections of MCM modules of a given rank on H is finite. In particular, the set of representatives of MCM modules of a fixed rank in $\overline{\mathrm{G}}(R)$ is finite.

Corollary

Let R be a local hypersurface with isolated singularity and $\dim R = 3$. Assume that $\operatorname{Spec} R$ admits a resolution of singularity. The set of MCM elements in the class group of R is finite. In particular, the set of ACM line bundles on a smooth surfaces in $\mathbb{P}^3$ is finite.

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Other applications:

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Other applications: Lower bounds for the betti numbers, length of socle, etc of system of ideals whose associated cycles has a limit that is non-zero in $\overline{\mathrm{G}}(R)$: I^n, $I^{(n)}$, $I^{[q]}$.