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Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Gennady Lyubeznik

Talk Title: Recent results on the grading of local cohomology modules

Date: 05/06/2013 **Time:** 11:00 (am) / pm (circle one)

List 6-12 key words for the talk: _____

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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Recent results on the grading of local cohomology modules

Gennady Lyubeznik
University of Minnesota

After a brief introduction to local cohomology I am going to discuss the result of my student Yi Zhang on the grading of local cohomology modules in characteristic p>0 and its recent extension to characteristic 0 by Linquan Ma and Wenliang Zhang. Namely, if R is a polynomial ring in n variables over a field and $m\subset R$ is the maximal ideal generated by the variables then it is well-known that $H^n_m(R)$ with its natural grading is isomorphic to $E(n)$, i.e. the naturally graded injective hull E of R/m degree-shifted downward by n. It has also been well-known that if $I\subset R$ is any ideal, then the local cohomology module $H^i_m(H^j_I(R))$ is isomorphic to a direct sum of a finite number of copies of E. Yi, Linquan and Wenliang sharpened this result by showing that if $I\subset R$ is any homogeneous ideal, then the local cohomology module $H^i_m(H^j_I(R))$, with its natural grading is isomorphic to a direct sum of a finite number of copies of $E(n)$. Some other related recent results will also be discussed.

Lyubeznik – Recent results on the grading of local cohomology modules

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R comm. rg, $I = (f_1, \dots, f_s)$ ideal

M R-module

$\check{C}(M; \underline{f})$ the following:

$$0 \to M \xrightarrow{d_0} \bigoplus_i M_{f_i} \xrightarrow{d_1} \bigoplus_{i<j} M_{f_i f_j} \xrightarrow{d_2} \cdots \xrightarrow{d_{s-1}} M_{f_1 \cdots f_s} \to 0$$

$$0 \qquad 1 \qquad\qquad 2 \qquad\qquad\qquad s$$

$$d_k(u)_{i_0 \dots i_k} = \sum_{j=0}^{k} (-1)^j u_{i_0 \dots \hat{i_j} \dots i_k}$$

where $u_{i_0 \dots \hat{i_j} \dots i_k} \in M_{f_{i_0} \cdots \hat{f_{i_j}} \cdots f_{i_k}}$

Def $H^i_I(M) := H^i(\check{C}(M; f_1, \dots, f_s))$

f_{i_0} is not f_j, as a rule

Depend only on M and $\sqrt{I}$

Ex $R = k[x_0, \dots, x_n]$, k field, $m = (x_0, \dots, x_n)$

Then $H^n_m(R) = R_{x_0 \cdots x_n} / \sum_{i=0}^{n} R_{x_0 \cdots \hat{x_i} \cdots x_n} = \sum_{\text{all simple}} \frac{k}{x_0^{i_0} \cdots x_n^{i_n}} \neq 0$

Supp $H = \{m\}$

$I \subset R = k[x_1, \dots, x_n] \Rightarrow \operatorname{Supp}(H^i_m(H^i_I(R))) = \{m\}$

Theorem: $H^j_m(H^i_I(R)) \cong H^{\oplus t}$ for some integer t

Q: [I homog, $V = V(I) \subseteq \mathbb{P}^{n-1}$. Does t only depend on V, j, and $n-i$?]

A: • Yes, if $\operatorname{char} k = p > 0$ (W. Zhang 2011)

• Yes, if V is smooth (N. Switala). Otherwise open in char 0

I homog, $f_1, \dots, f_s$ homog, M graded

$\Rightarrow H^i_I(M)$ acquire a natural grading

Ex: $\deg\left(\frac{1}{x_1^{i_1} \cdots x_n^{i_n}}\right) = -i_1 - i_2 - \cdots - i_n$

Def with this grading $H = {}^*H$.

Theorem: I homog. $\Rightarrow H^j_m(H^i_I(R)) \cong^{(\text{gradedly})} {}^*H^{\oplus t}$

(Y. Zhang (char p, 2011), L. Ma & W. Zhang (char 0, arXiv))

Consequence: $R = k[x_1, \dots, x_n]$, $V \subset \mathbb{P}^{n-1}_k$, $I = I(V) \subseteq R$.

$\Rightarrow (H^i_I(R))_\delta \cong H^{i-1}(\mathbb{P}^{n-1} \setminus V, \mathcal{O}_{\mathbb{P}^{n-1}}(\delta))$, for any $\delta \in \mathbb{Z}$, $i \geq 2$

also: V smooth $\Rightarrow \operatorname{supp} H^i_I(R) = \{m\}$ $\forall i > c = \operatorname{codim} V$

$\Rightarrow H^0_{\mathfrak{m}}(H^i_I(R)) = H^i_I(R)$ $\forall i > c$

$\cong [{}^*H^{\delta}]_0$

~~If~~ ${}^*H_\delta = 0$ $\forall \delta > -n-1$

$\Rightarrow H^i(\mathbb{P}^n \setminus V; \mathcal{O}(\delta)) = 0$ $\forall \delta > -n-1, i > c$.

Recall $H^i_I(R) = \varinjlim_t \operatorname{Ext}^i_R(R/I^t, R)$

; Maps are induced by the natural maps $R/I^{t+1} \to R/I^t$

Work in progress (Bhatt, Blickle, L., Smith, W. Zhang)

Let $V \subseteq \mathbb{P}^n_k$ smooth, $\operatorname{codim} c$, $R = k[x_0, \dots, x_n]$,

$I = I(V)$, $i > c$. Assume $\operatorname{char} k = 0$.

Then for every δ, the map $\operatorname{Ext}^i_R(R/I^t, R)_\delta \to H^i_I(R)_\delta$

is also 0, in degrees $\geq \delta$ for $t \gg 0$

Cor: $\operatorname{Ext}^i_R(R/I^t, R)_\delta = 0$ for $\delta > -n-1$, for $t \gg 0$

Open question: $V \subset \mathbb{P}^n_k$, any projective scheme.

Is it true that $\forall i$, $H^0_m(\mathrm{Ext}^i_R(R/I^t, R)) \xrightarrow{\cong} H^0_m(H^i_I(R))$ for $t \gg 0$?

Open question: $I \subset R$ arbitrary. Is it true that for $t \gg 0$, $\mathrm{Ext}^i_R(R/I^t, R) \longrightarrow H^i_I(R)$ induces an isom. on socles?

Ma-Zhang: $H^i_m(H^j_I(R)) \cong H^{d-t}_H$, if $\mathrm{char}\, k = 0$.

Key idea: Let $D := R\langle d_1, \ldots, d_n \rangle$, where $d_i = \frac{\partial}{\partial x_i} : R \to R$.

~~This is a left D-m~~

Note that R, R_f, $H^i_I(R)$ are D-modules.

Def: A graded D-module M is Eulerian if

$$(x_1 d_1 + \cdots + x_n d_n)(m) = (\deg(m)) \cdot m.$$

e.g. R is Eulerian (due to Euler!)

e.g. If M is Eulerian, so are M_f, $H^i_I(R)$, and $H^j_m(H^i_I(R))$.