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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Claudia Polini

Talk Title: The core of an ideal

Date: 05 / 06 / 2013 **Time:** 2:00 am / pm (circle one)

List 6-12 key words for the talk: ____________________

Please summarize the lecture in 5 or fewer sentences: The speaker motivates, explains, and gives examples of the notion of the "core" of an ideal in a commutative Noetherian ring. Formulas and properties are explored as well.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

C. Polini — The core of an ideal:

R standard graded k-algebra, k a field, R is CM, $d = \dim R$

$\tilde{R} := R/(l_1, \dots, l_d)$, where $l_1, \dots, l_d$ are general linear forms. Let $0 \neq \alpha \in R$.

Q: Does there exist $\tilde{R}$ s.t. $0 \neq \tilde{\alpha} \in \tilde{R}$

$(\Leftrightarrow \alpha \notin \bigcap (\underline{l}))$

Leads to the question: What is $\bigcap(\underline{l})$?

Note: $\mathfrak{m}^{\operatorname{reg}(R)+1} \subseteq \bigcap(\underline{l})$.

Q: When is there an equality?

(Yes if R is Gor. or R is a domain)

R Noetherian, $J \subseteq I$ is a reduction

$\Leftrightarrow \underbrace{I^{n+1} = JI^n}$ for some n

$\Leftrightarrow$ " for $n \gg 0$

$\Leftrightarrow R[Jt] \subseteq R[It]$ is integral (module finite)

Def $\text{core}(I) = \bigcap_{J \text{ min red. of } I} J$ (Rees & Sally)

Note: $\text{core}(I) \subset I \subset \sqrt{\text{core}(I)}$

Note: In general, $\text{core}(I)$ is not a reduction of I.

From now on: (R, m, k) local CM, $|k| = \infty$, $\sqrt{I} = m$
($\underline{\text{Note}}$: $\sqrt{I} = m$ isn't quite necessary)

Note: All the min reductions of I are d-generated

Example: $J \subseteq I \subseteq k[x_1, \ldots, x_d]$ monomial ideals

J red. of $I \Leftrightarrow \underset{\substack{\text{Newton} \\ \text{polyhedron}}}{NP}(I) = NP(J)$

Rees-Sally (1988), Huneke-Swanson (1995),
Corso-P.-Ulrich, Hyry-Smith, P.-Ulrich, Huneke-Trung,
Fouli-P.-Ulrich.

Briançon-Skoda Thm: R reg. loc ring $\Rightarrow I^d \subseteq \text{core}(I)$,

Lipman: $I^d \subseteq \text{adj}(I^d) \subseteq \text{core}(I)$. (1994)

Def (Lipman) ~~I a~~ ~~ideal~~ ~~V~~ ~~a ... ideal of R.~~

$\mathrm{adj}(I) = \bigcap_V I\,\mathcal{O}_{V}:_{K} \omega_{V/R}$, intersection taken over all DVRs V ~~...~~ e.o.f.t./R s.t. $R \subset V \subset \mathrm{Quot}(R)$

(Note: $\bar{I} \subseteq \mathrm{adj}(I)$).

Question: When does $\mathrm{core}(I) = \mathrm{adj}(I^d)$?

H-S: Kawamata conjecture.

~~Question~~.

Thm [Polini-Ulrich]:

(a) char $k = 0$, R is Gorenstein, normal.

~~then~~ $\mathrm{Proj}(R[It])$ is R_1 $\Rightarrow$ $\mathrm{core}(I)$ is int. closed.

(b) R RLR, e.o.f.t. over fld of char 0, $R[It]$ has rat'l sings

$\Rightarrow \mathrm{core}(I) = \mathrm{adj}(I^d)$.

Thm [Hyry]: R e.f.t. RLR $R[It]$ normal, CM, ~~...~~

If $\mathrm{core}(I) = \mathrm{adj}(I^d)$ then $R[It]$ has rat'l sings

Ex: * [Kohlhaas]: $I \subset k[x_1, \dots, x_d]$ monomial ideal w/ a d-generated monomial red.

* [Kustin-P.-Ulrich]: $I \subset k[x,y]$, I gen by forms of the same degree.

Then: $\operatorname{core}(I) = \operatorname{adj}(I^d) \Leftrightarrow \operatorname{core}(I)$ is int. closed $\Leftrightarrow R[It]$ is (R_1).

Thm [Corso-P.-Ulrich]:

$$\operatorname{core}(I) = \bigcap_{\text{finite}} \text{"general minimal reductions"}$$

[This is true as long as I is G_ℓ and $(\ell-1)$-residually S_2, where $\ell = \ell(I)$, and (R, m, k) Noetherian with $|k| = \infty$].

Formula: (Conj. by CPU, pf in Hyry-Smith, extended in PU, HT, FPU)

char $k = 0$ or char $k \gg 0$, or I gen. by forms of the same degree, and R is geometrically reduced.

$\left(= (\omega_{R[It]})_d\right)$

Then $\operatorname{core}(I) = (J^{n+1} : I^n)$, $n \gg 0$ where J a min. reduction of I ($n \geq$ reduction # of I)

(e.g. if R is geom. reduced then

[Fouli-P.-Ulrich] $m^{\operatorname{reg}(R)+1} \subset \operatorname{core}(m) = (\underline{\ell})^{\operatorname{reg}(R)+1} : m^{\operatorname{reg} R}$

with equality $\iff ([\omega_R]_{d-\operatorname{reg}(R)})\, R$ is faithful as an R-module.)

Ex: $X = \{p_1, \ldots, p_e\}$ reduced points in $\mathbb{P}^n$

m = homog. max. ideal of $A(X)$

X is Cayley-Bacharach $\iff \operatorname{core}(m) = m^{\operatorname{reg}(R)+1}$

Ex:  is not C-B, and indeed $m^3 \subset \operatorname{core}(m) = m^3 + (x^2)$.

The monomial case: Let $I \subset k[x_1, \ldots, x_d]$,
$I = (f_1, \ldots, f_n)$, f_i monomials
$\sqrt{I} = m$

Fact: char $k = 0$, $\overline{I^j} = I^j$ for $j \gg 0$ $\Rightarrow$ core$(I) = $ adj(I^d)

Note: Here the core of I is a monomial ideal
(since it is stable under the torus action on $k[x_1, \ldots, x_d]$)

For any $L \subset k[x_1, \ldots, x_d]$ ideal,

mono(L) = largest monomial ideal contained in L

Hence: core$(I) \subseteq$ mono(J) $\forall$ min. reductions J of I

Let H = (d general ~~lo~~ linear ~~com~~ combinations of the f_i).

Let $f \in I$ and $J = (H, f^d)$. This is a "general locally minimal reduction of I".

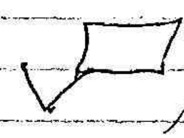
Thm: Then $\mathrm{core}(I) = \mathrm{mono}(J)$

Pf: $\mathrm{mono}(J)$ does not depend on such general J.

We know that $\mathrm{core}(I) = J_1 \cap \cdots \cap J_t$, J_i general.

then $\mathrm{mono}(J) = \mathrm{mono}(J_i) \subseteq J_i$

So $\mathrm{mono}(J) = \bigcap_i \mathrm{mono}(J_i) \subseteq \bigcap_i J_i = \mathrm{core}(I)$ $\square$

Ex: [Fouli-Morey]: Take the edge ideal of ,

or [P-U]. $I = (x^3, x^2y, xz^2, z^3)$

Then $\mathrm{core}(I) \neq \mathrm{mono}(J)$ for a general J.

Indeed, $\mathrm{core}(I) \neq J_1 \cap \cdots \cap J_t$, J_i gen. reds

$B = (\sum_j z_{ij} f_j \mid 1 \leq i \leq d) \subseteq k[\underline{z}][\underline{x}]$

$\mathrm{mono}(B) = \mathrm{mono}_1(B) = c_1 M_1 + \cdots + c_t M_t$

C_i ideals in $k[\underline{z}]$

M_i monomials in $k[\underline{x}]$

Thm: $(M_1, \ldots, M_3) \subseteq \mathrm{core}(I) \subset \mathrm{mono}(J) = (M_1, \ldots, M_t)$
(J gen.)

$\mathrm{Content}(\mathrm{mono}(\mathcal{B})) = C_1 + \cdots + C_t = (F)$

determines the locus of $\underline{\lambda}$ s.t. $\underline{\lambda} \cdot \mathcal{B}_{\underline{\lambda}}$ is a reduction of $\overline{I}$

Conj: $\sqrt{C_i} = (F)$, $1 \leq i \leq s$