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Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Steven Dale Cutkosky

Talk Title: Multiplicities of graded families of linear series

Date: 05 / 07 / 2013 **Time:** 9 : 00 am / pm (circle one)

List 6-12 key words for the talk: ______

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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Multiplicities of graded families of linear series

Steven Cutkosky
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We give simple necessary and sufficient conditions on projective schemes over a field k for asymptotic limits of the growth of all graded linear series of a fixed Kodaira-Iitaka dimension to exist. We begin by defining the volume of line bundles and graded linear series, cover some interesting properties, and discuss methods for computing it.

Multiplicities of graded families of linear series.

Assumption (*): X is a d-dimensional proj. scheme / k, a field.

$$\mathrm{Vol}(\mathcal{L}) := \limsup_{n\to\infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d/d!}$$

($\mathcal{L}$ a line bdl.)

Motivation: If $\mathcal{L}$ is ample, $\mathrm{vol}(\mathcal{L}) = (\mathcal{L}^d)$ [i.e. its self-$\cap$ number]

Note: $\mathrm{Vol}(\mathcal{L})$ can be irrational (joint w/ Srinivas), even on a nonsingular variety / $\mathbb{C}$.

Thm (Fujita, Lazarsfeld, ...) Suppose X is a proj. var. (i.e. integral proj. scheme) over an alg. closed field k, and $\mathcal{L}$ is a line bundle on X. Then: $\mathrm{Vol}(\mathcal{L}) = \lim_{n\to\infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d/d!}$.

Proofs: In char 0, Lazarsfeld, Fujita approx (needs resol. of singularities)
Satoshi, Takagi in char $p>0$, De Jong using Fujita approx

Core method
Okanka
Lazarsfeld-Mustata

$\mathcal{L}$ a line bundle on X.

Assuming (*), a graded linear series L on X is a graded k-subalgebra.
indeed $L = \bigoplus_{n\geq 0} L_n \subset \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^n)$ as a graded k-subalgebra.

Index: $m(L) = [\mathbb{Z} : G]$, $G = \langle n \mid L_n \neq 0 \rangle$. (Assume for now that $m(\mathcal{L}) = 1$)

For the method to work, you need
(**) 1. X is integral.
2. $\exists$ a k-rational pt Q on X that is nonsingular.

Associate a semigroup $S(L) \subset \mathbb{N}^{d+1}$ to L.
$S(L)_n = S(L) \cap (\mathbb{N}^d \times \{n\})$.
$\#(S(L)_n) = \dim_k L_n$.
$S(L)$ is bounded.

general context ($k \neq \bar{k}$)

Thm: (Okounkov, Lazarsfeld-Mustata):
$\lim_{n\to\infty} \frac{\# S(L)_n}{n^d}$ exists.
It is proportional to the volume of:
$\Delta(S(L)) = (\text{real cone gen. by } S(L)) \cap (\mathbb{R}^d \times \{1\})$.

(basic idea goes back to Minkowski)

Can prove limits if X is geometrically integral.

$X \otimes_k \bar{k}$ is a variety

$\dim_k L_n = \dim_{\bar{k}} (L_n \otimes_k \bar{k})$.

Example (—): $\exists$ a proj scheme X over any field k that is irreducible but not reduced and a graded linear ser. L on X s.t.

$$\lim_{n\to\infty} \frac{\dim_k L_n}{n^d} \text{ does not exist,}$$

even if n is constrained to lie in any arithmetic sequence.

Related examples for non-reduced local rings (by H. Dao & I. Smirnov)

$\exists$ proj. varieties X s.t. $X \otimes_k \bar{k}$ is not integral, nor even reduced, if k is not perfect.

Assumption (2) can be removed using semigroups.

Kodaira-Iitaka dimension.

$$\sigma(L) := \max \{ s \mid \exists y_1, \ldots, y_s \in L, \text{ homog. of positive degree and alg. indep.}/k \}$$

Then set $K(L) := \begin{cases} \sigma(L)-1 & \text{if } \sigma(L) > 0, \\ -\infty & \text{if } \sigma(L) = 0. \end{cases}$

(classical if X is a normal variety)

Thm (Kaveh-Khovanskii)

Suppose X is proper var. / alg. closed field k, L a graded lin. ser., $m := m(L) =$ index.
Then

$$\lim_{n\to\infty} \frac{\dim_k L_{mn}}{n^{K(L)}} \text{ exists.}$$

$(m \nmid n \Rightarrow L_{mn} = 0)$

Thm (–): Suppose X is a proj. scheme / a field k.

$N_X := \{ f \in \mathcal{O}_X \mid f^n = 0 \text{ for some } n > 0 \}$

Suppose $\alpha \in \mathbb{N}$. TFAE:

1. For every graded linear-series L on X with $K(L) \geq \alpha$. $\exists r \in \mathbb{Z}_{>0}$ such that
$$\lim_{a \to \infty} \frac{\dim_k L_{ar+b}}{a^{K(L)}} \text{ exists } \forall b \in \mathbb{N}.$$
2. For every GLS L with $K(L) \geq \alpha$, $\exists r, b$ s.t.
$$\lim_{a \to \infty} \frac{\dim_k L_{ar+b}}{a^{K(L)}} \text{ exists.}$$
3. $\dim \operatorname{Supp}(N_X < \alpha)$.

$3 \Rightarrow 1$ is true for proper schemes, compact complex analyt. spaces.

Cor: Suppose X is a d-dim'l proper scheme / field k with $\dim N_X < d$, $\mathcal{L}$ a line bdl. on X. Then $\operatorname{Vol}(\mathcal{L}) = \lim_{n \to \infty} \dfrac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d / d!}$.

Example: k any field, $X = \operatorname{proj}(k[X_0, X_1, X_2]/(X_1^2) = k[\bar{X}_0, \bar{X}_1, \bar{X}_2])$, and T any infinite subset of $\mathbb{Z}_{>0}$ with $\mathbb{Z}_{>0} \setminus T$ infinite.

Set $L_n :=$ The k-subspace of $\Gamma(X, \mathcal{O}_X(n))$ with basis
$\bar{X}_0^{n-1}\bar{X}_1, \bar{X}_0^{n-2}\bar{X}_1\bar{X}_2, \ldots, \bar{X}_0^{n-\lambda(n)}\bar{X}_1\bar{X}_2^{\lambda(n)-1}$, where
$$\lambda(n) = \begin{cases} \lceil \log(n) \rceil & \text{if } n \in T \\ \lceil \log(n)/2 \rceil & \text{if } n \notin T. \end{cases}$$

Then $L_m L_n = 0$ $\forall m, n > 0$, $\dim_k L_n = \lambda(n)$, $K(L) = -\infty$.

With a little care, can extend to a $K(L) = d$ $(=1)$ example.