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Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Vasudevan Srinivas

Talk Title: Ordinary varieties and the comparison between multiplier ideals and test ideals

Date: 05 / 07 / 2013 **Time:** 10 : 30 am / pm (circle one)

List 6-12 key words for the talk: ____________________

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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Ordinary varieties and the comparison between multiplier ideals and test ideals

Vasudevan Srinivas
Tata Institute

This talk will review joint work with M. Mustata. This relates a conjecture on ordinary reduction to positive characteristics of smooth proper varieties in characteristic 0, and reduction of multiplier ideals in characteristic 0 to test ideals in postive characteristics.

Ordinary varieties and the comparison between multiplier ideals and test ideals
(joint with M. Mustaţă)

X: variety defined $/k$ = field of char 0, $k = \bar{k}$

"model": $A \subset k$ finitely generated $\mathbb{Z}$-algebra, $X_A \to \operatorname{Spec} A$, whose base change to k is X.

$s \in \operatorname{Spec} A$ closed pt: $X_s \longrightarrow \operatorname{Spec} \underbrace{k(s)}_{\text{finite field}}$

<u>Thm</u>: X has rational singularities $\Leftrightarrow$ $\exists$ dense open $U \subset \operatorname{Spec} A$ s.t. $\forall$ closed points $s \in U$, X_s has "F-rational" singularities.

($\Leftarrow$ by Karen Smith; $\Rightarrow$ by N. Hara; Mehta-.)

Singularities — min. model program

Hara & Watanabe

log canonical sings $\longleftrightarrow$ char 0

F-purity $\longleftrightarrow$ char $p > 0$

If $\exists$ a dense set of closed pts $s \in \operatorname{Spec} A$ s.t. X_s is F-pure, then X has log canonical singularities

— converse is conjectural.

<u>Example</u>: X = cone over an elliptic curve E (char 0)

— log canonical

— $X_s \leftrightarrow E_s$

$\underset{\text{(F-pure)}}{X_s \text{ is F-split}} \Leftrightarrow E_s$ is an <u>ordinary</u> elliptic curve.

<u>Def</u>: E/F = field of char p, $F = \bar{F}$, $E[p]$ = p-torsion pts of E. We know that $E[p] = \mathbb{Z}/p\mathbb{Z}$ or 0.

"E is ordinary" ($\mathbb{Z}/p\mathbb{Z}$) — "super singular" (0)

Y smooth proj. (proper) variety $/k$ = perfect field of char $p > 0$. $F: Y \to Y$ the Frobenius map

F_* (de Rham complex):

$$0 \to F_* \mathcal{O}_Y \xrightarrow{d} F_* \Omega^1_{Y/k} \xrightarrow{d} \cdots \xrightarrow{d} F_* \Omega^d_{Y/k} \to 0$$

$f \cdot \omega = f^p \omega$, $d(f^p \omega) = f^p d\omega = f \cdot d\omega$

$B^i_{Y/k} = \operatorname{im}(F_* \Omega^{i-1}_{Y/k} \xrightarrow{d} F_* \Omega^i_{Y/k})$

$Z^i_{Y/k} = \ker(F_* \Omega^i_{Y/k} \xrightarrow{d} F_* \Omega^{i+1}_{Y/k})$.

$$0 \to B^i_{Y/k} \to Z^i_{Y/k} \to \mathcal{H}^i_{DR}(Y/k) \to 0, \quad \mathcal{H}^i_{DR}(Y/k) \cong \Omega^i_{Y/k}$$

Y smooth $\Rightarrow$ $\exists$ <u>Cartier operator</u> (isomorphism). $C(f^p \omega) = f C(\omega)$, $C(\omega_1 \wedge \omega_2) = C(\omega_1) \wedge C(\omega_2)$, $C(d\omega) = 0$, $C(d \log f) = d \log f$

Def: Y is called ordinary in the sense of Bloch-Kato (+Deligne, Illusie, ...) if $H^i(Y, B^j_{Y/k}) = 0$ $\forall i, j$.

Conjecture: If $X/k = \bar{k}$ (char 0) is smooth & projective, $X_A \to \operatorname{Spec} A$ model, then $\exists$ dense set of closed pts $s \in \operatorname{Spec} A$ s.t. $X_s / \operatorname{Spec} k(s)$ is ordinary.

- If $Y/k = \bar{k}$ in char p is an abelian variety of smooth proj curve, then this is the usual notion of "ordinarity"

(A = abelian var $\Rightarrow A[p](\bar{k}) \cong (\mathbb{Z}/p\mathbb{Z})^r$ for some r with $0 \le r \le \dim A$,
$r = \dim A \Leftrightarrow A$ is an "ordinary Abelian variety" (in any sense))

Ogus: Conj holds for abelian varieties of dim ≤ 2

Let $f: Y \to S$ be a family in char p, $/k = \bar{k}$. $\{s \in S \mid s \text{ closed}, Y_s \text{ ordinary}\}$ is open in S.
s closed pt.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_Y \arrow[r] & F_*\mathcal{O}_Y \arrow[r] \arrow[dr, "d"'] & B^1_{Y/k} \arrow[r] \arrow[d, phantom, "\subseteq" rotate=-90] & 0 \\ & & & F_*\Omega^1_{Y/k} & \end{tikzcd}$$

If Y is ordinary, then $F: H^i(Y, \mathcal{O}_Y) \to H^i(Y, \mathcal{O}_Y)$ is bijective $\forall i$

Weak ordinarity Conjecture: If $X_A \to \operatorname{Spec} A$ is a model of a smooth proper variety X/k in char 0, then $\exists$ a dense set $s \in \operatorname{Spec} A$ of closed pts. s.t. Frobenius is injective on $H^{\dim X_s}(\mathcal{O}_{X_s})$

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multiplier ideals (char 0) $\longleftrightarrow$ test ideals (char $p > 0$) (Hara + Yoshida)

[In both cases, let Y/k irred smooth variety, construct $\mathfrak{a} \subset \mathcal{O}_Y$ ideal]

$k = \bar{k}$ char 0.
$\pi: X \to Y$ log resolution:
i) $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-G)$, G effective divisor, and
ii) $\exists$ simple normal crossing div. E on X s.t. both G and $K_{X/Y}$ are supported on E.

The multiplier ideal of exponent $\lambda \in \mathbb{R}_{\ge 0}$ is: $\mathcal{J}(Y, \mathfrak{a}^\lambda) = \pi_*\mathcal{O}_X(K_{X/Y} - \lfloor \lambda G \rfloor)$.

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k: char $p > 0$, $0 \to B^n_{Y/k} \to F_*\omega_Y \xrightarrow{C} \omega_{Y/k} \to 0$. We have $C^e: F^e_*\omega_Y \to \omega_Y$.
(idea: C^e is like taking p^eth roots).
So if $\mathfrak{b} \subset \mathcal{O}_Y$ ideal, $\mathfrak{b}^{[1/p^e]}$ is defined by $C^e(\mathfrak{b} \cdot \omega_Y) = \mathfrak{b}^{[1/p^e]} \cdot \omega_Y$.

$(\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]} \subset (\mathfrak{a}^{\lceil \lambda p^{e+1} \rceil})^{[1/p^{e+1}]}$, $\forall e$. Hence we get an ascending chain of ideals in $\mathcal{O}_Y$.

Def: The test ideal with parameter λ is $\tau(Y, \mathfrak{a}^\lambda) :=$ the stable value of this chain.

Thm: The weak ordinarity conjecture $\Rightarrow$ for $(X, \mathfrak{a})$ over k in char 0, and a model $X_A \to \operatorname{Spec} A$, $\mathcal{J}(X, \mathfrak{a}^\lambda)_s = \tau(X_s, \mathfrak{a}_s^\lambda)$ for a dense set of closed points $s \in \operatorname{Spec} A$, $\forall \lambda \ge 0$.

log canonical threshold $= \sup\{\lambda \mid \mathcal{J}(X, \mathfrak{a}^\lambda) = \mathcal{O}_X\}$.
F-pure threshold $= \sup\{\lambda \mid \tau(X_s, \mathfrak{a}_s^\lambda) = \mathcal{O}_{X_s}\}$