

## NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Neil Epstein Email/Phone: nepstei2@gmu.edu

Speaker's Name: Karl Schwede

Talk Title: F-singularities in families

Date: 05/07/2013 Time: 11:30 (a) / pm (circle one)

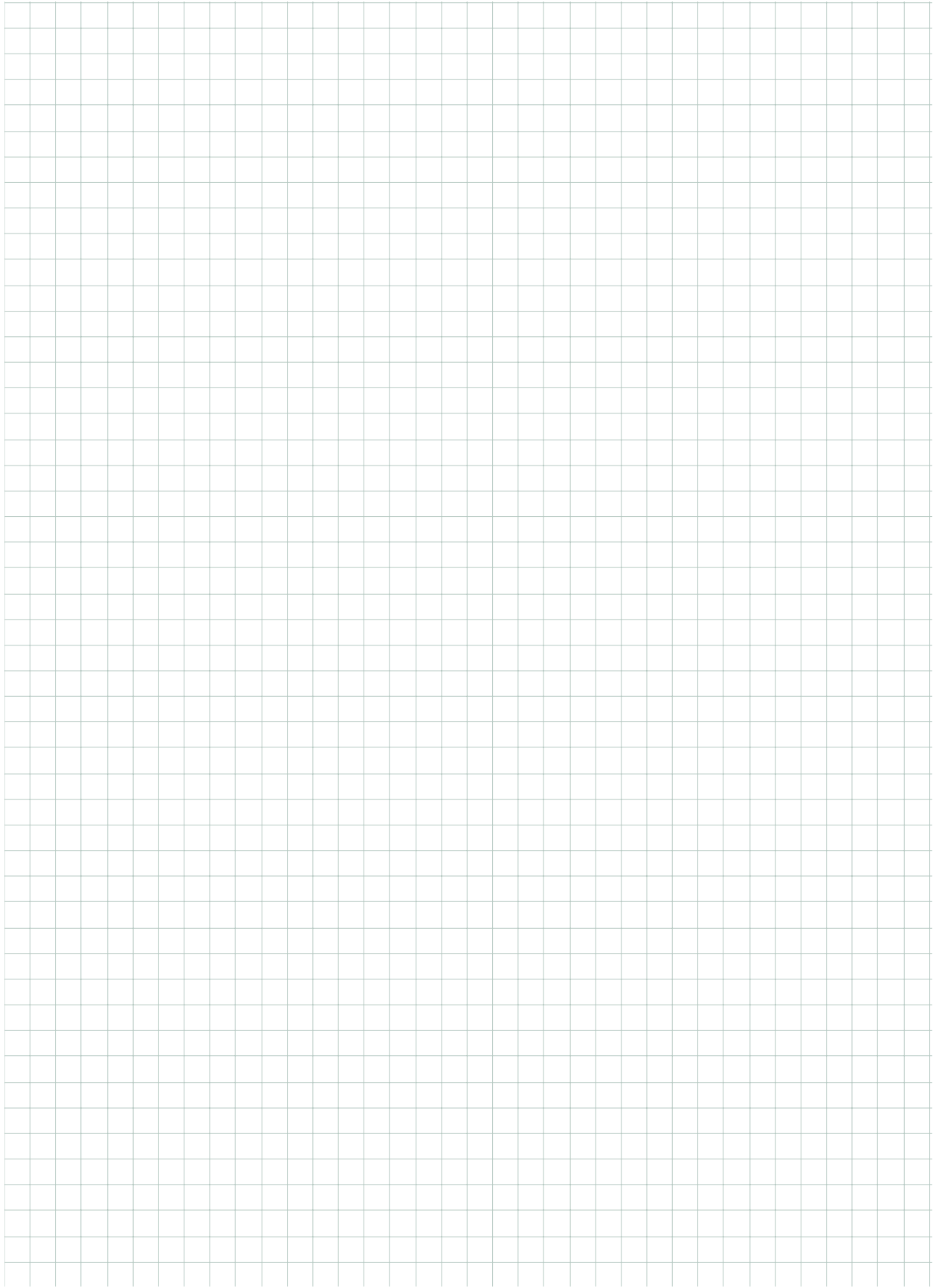
List 6-12 key words for the talk: \_\_\_\_\_

Please summarize the lecture in 5 or fewer sentences: (see abstract)

## CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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(YYYY.MM.DD.TIME.SpeakerLastName)
- Email the re-named files to [notes@msri.org](mailto:notes@msri.org) with the workshop name and your name in the subject line.



# **F-singularities in families**

***Karl Schwede***

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F-singularities are classes of singularities defined by the behavior Frobenius. A prominent tool for measuring these singularities is the test ideal, a characteristic  $p > 0$  analog of the multiplier ideal. Recently, there has been interest in applying the methods of F-singularities to a number of geometric problems in positive characteristic. However, one gap in the theory has been the behavior of F-singularities in families. For example restriction theorems for test ideals have been lacking. In this talk, I will discuss recent joint work with Zsolt Patakfalvi and Wenliang Zhang on the behavior of F-singularities and test ideals in families. For example, we will obtain generic (and non-generic) restrictions theorems for test ideals. Some global geometric consequences will also be discussed if there is sufficient time.

# F-singularities in families

(joint w/ Zolt Patakfalvi & Wenliang Zhang)

Motivation in char 0: Kodaira (or Kawamata-Viehweg) vanishing.

Thm:  $X$  smooth /  $\mathbb{C}$ ,  $L$  ample (pres. big & nef)  $H^i(X, \omega_X \otimes L) = 0 \forall i > 0$

note: This is false for singular  $X$ , but it holds for rational singularities.

Def:  $X$  has rational singularities if ①  $X$  is Cohen-Macaulay, and ② If  $\pi: Y \rightarrow X$  is a resolution, then  $\pi_* \mathcal{O}_Y = \mathcal{O}_X$ .

Aside:  $R^i \pi_* \omega_Y = 0 \forall i > 0$ . Also  $\pi_* \mathcal{O}_Y \subset \mathcal{O}_X$ .

Thm: (Kodaira holds for rat'l singularities):

$$\text{Eg: } H^i(\omega_X \otimes L) = H^i(\pi_* \omega_Y \otimes L) = H^i(Y, \omega_Y \otimes \pi^* L) = 0.$$

Observe that  $H^i(X, \pi_* \omega_Y \otimes L) = 0 \forall i > 0$ .

(Recall that  $\pi_* \omega_Y$  is indep. of the resolution)

If  $X$  is Gorenstein,  $\pi_* \omega_Y = \mathcal{J}(X) \omega_X$

## Motivation in char $p > 0$ :

note: Kodaira vanishing is false in char  $p > 0$ , even for smooth  $X$ .

Given  $(X, L)$ , <sup>ample</sup> pull back, get vanishing.

Let  $F: X \rightarrow X$  denote the absolute Frobenius. Then  $F^* L = L^p$ . (Also, can iterate to  $F^e: X \rightarrow X$ ,  $F^{e*} L = L^{p^e}$ )

We also have:  $H^i(X, \omega_X \otimes F^{e*} L) = H^i(X, \omega_X \otimes L^{p^e}) = 0 \forall i > 0$ , by Serre vanishing.

We have  $F_*^e \omega_X \rightarrow \omega_X$ . Which to use?

$$F_*^{e+1} \omega_X \rightarrow F_*^e \omega_X \rightarrow \dots \rightarrow F_* \omega_X \xrightarrow{\mathbb{I}} \omega_X$$

Fact: These images stabilize for  $e \gg 0$ .  
(Hartshorne-Speiser, Lyubeznik, Gabber, Blickle)

Define the stable image (when  $X$  is Gorenstein) to be  $\sigma(X) \omega_X$ .

note:  $\sigma(X)$  is a non-F-pure ideal. It is a variant of a test ideal.

Given a flat family  $f: X \rightarrow V$ ,  $V$  a smooth curve, Gorenstein fibers, geometrically reduced fibers <sup>finite type</sup>. How do  $\sigma(X)$  and  $\sigma(X_s)$  compare for  $s \in V$  a closed point? (separated,  $G, e, S_2$ )

$\sigma(X)$  and  $\sigma(X_s)$  compare for  $s \in V$  a closed point?

In char 0,  $\mathcal{J}(X) \cdot \mathcal{O}_{X_s} = \mathcal{J}(X_s) \forall s \in U \subseteq V$  a dense open set.

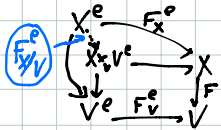
Example (Mustata-Yoshida): let  $V = \text{Spec } k[t]$ ,  $X = \text{Spec } k[x, t]$ ,  $D = \text{div}_X(x^p - t)$ .

Consider  $\sigma(X, D)$  vs  $\sigma(X_s, D_s)$ .

$$\mathcal{O}_X \neq \sigma(X_s, p \cdot \text{div}_X(x - t^p))$$

$$\begin{array}{ccc} \frac{F_X \omega_X}{t} & \xrightarrow{\quad} & \frac{\omega_X}{t}, \text{ consider mod } t=0 \\ \parallel & & \uparrow \text{?} \\ F_X(\omega_X/t) & \xleftarrow{\quad} & F_X(\omega_X/t) \end{array}$$

Relative Frobenius map (Rohrbaugh-Andre map)



If  $s \in V$  is a point with perfect residue field, relative & absolute Frobenius coincide

We look at  $F_X^e \omega_X \xrightarrow{\star} \omega_{X_{x_v, V^e}}$ ; this behaves well with respect to base change.

Define  $\sigma_e(X/V) \cdot \omega_{X_{x_v, V^e}}$  to be the image of  $\star$ .

Thm:  $\exists e_0 > 0$  s.t.  $\forall e \geq e_0$ ,  $\sigma_e(X/V) \cdot \mathcal{O}_{V/m_s} = \sigma(X_s)$  for geometric  $s \in V$ .

Corollary:  $\exists e_0 > 0$  s.t.  $\forall e \geq e_0$ ,  $\sigma(X_{x_v, V^e}) \cdot \mathcal{O}_{X_{x_v, V^e}/m_s} = \sigma(X_s)$ ,  $\forall s \in V$ .

note:  $\tau(X_{x_v, V^e}) \cdot \mathcal{O}_{X_{x_v, V^e}/m_s} = \tau(X_s)$   $\forall s \in U$  (open)

$$\sigma_e(X/V) \otimes_{\mathcal{O}_V} \mathcal{O}_{V/m_s} \cong \sigma_{e+1}(X/V)$$

