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Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Karl Schwede

Talk Title: F-singularities in families

Date: 05 / 07 / 2013 **Time:** 11 : 30 am / pm (circle one)

List 6-12 key words for the talk: ______

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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F-singularities in families

Karl Schwede
Pennsylvania State University

F-singularities are classes of singularities defined by the behavior Frobenius. A prominent tool for measuing these singularities is the test ideal, a characteristic $p > 0$ analog of the multiplier ideal. Recently, there has been interest in applying the methods of F-singularities to a number of geometric problems in positive characteristic. However, one gap in the theory has been the behavior of F-singularities in families. For example restriction theorems for test ideals have been lacking. In this talk, I will discuss recent joint work with Zsolt Patakfalvi and Wenliang Zhang on the behavior of F-singularities and test ideals in families. For example, we will obtain generic (and non-generic) restrictions theorems for test ideals. Some global geometric consequences will also be discussed if there is sufficient time.

F-singularities in families
(joint w/ Zsolt Patakfalvi & Wenliang Zhang)

Motivation in char 0: Kodaira (or Kawamata-Viehweg) vanishing.

Thm: X smooth$/\mathbb{C}$, L ample (resp. big & nef) $H^i(X, \omega_X \otimes L) = 0 \ \forall i > 0$

note: This is false for singular X, but it holds for rational singularities.

Def: X has rational singularities if ① X is Cohen-Macaulay, and ② If $\pi: Y \to X$ is a resolution, then $\pi_* \mathcal{O}_Y = \mathcal{O}_X$.

Aside: $R^i \pi_* \omega_Y = 0 \ \forall i > 0$. Also $\pi_* \mathcal{O}_Y \hookrightarrow \mathcal{O}_X$.

Thm: (Kodaira holds for rat'l singularities).

pf: $H^i(\omega_X \otimes L) = H^i((\pi_* \omega_Y) \otimes L) = H^i(Y, \omega_Y \otimes \pi^* L) = 0$.

Observe that $H^i(X, (\pi_* \omega_Y) \otimes L) = 0 \ \forall i > 0$.

(recall that $\pi_* \omega_Y$ is indep. of the resolution)

If X is Gorenstein, $\pi_* \omega_Y = \mathcal{J}(X) \cdot \omega_X$

Motivation in char $p > 0$:

note: Kodaira vanishing is false in char $p > 0$, even for smooth X.

ample

Given (X, L), pull back, get vanishing.

Let $F: X \to X$ denote the absolute Frobenius. Then $F^* L = L^p$. (Also, can iterate to $F^e: X \to X$, $F^{e*} L = L^{p^e}$.

We also have: $H^i(X, \omega_X \otimes F^{e*} L) = H^i(X, \omega_X \otimes L^{p^e}) = 0$ $e \gg 0$, by Serre vanishing.

We have $F^e_* \omega_X \to \omega_X$. Which e to use?

$$F^{e+1}_* \omega_X \to F^e_* \omega_X \to \cdots \to F_* \omega_X \xrightarrow{T} \omega_X$$

Fact: These images stabilize for $e \gg 0$.
(Hartshorne-Speiser, Lyubeznik, Gabber, Blickle)

Define the stable image (when X is Gorenstein) to be $\sigma(X) \cdot \omega_X$.

note: $\sigma(X)$ is a non-F-pure ideal. It's a variant of a test ideal.

Given a flat family $f: X \to V$, V a smooth curve, Gorenstein fibers (generalizes to G_1 & S_2), geometrically reduced fibers, finite type. How do $\sigma(X)$ and $\sigma(X_s)$ compare for $s \in V$ a closed point?

In char 0, $\mathcal{J}(X) \cdot \mathcal{O}_{X_s} = \mathcal{J}(X_s)$ $\forall s \in U \subseteq V$ a dense open set.

Example (Mustață-Yoshida): let $V = \mathrm{Spec}\, k[t]$, $X = \mathrm{Spec}\, k[x,t]$, $D = \mathrm{div}_X(x^p - t)$.

Consider $\sigma(X, D)$ vs. $\sigma(X_s, D_s)$.

$$\sigma(X,D) = \mathcal{O}_X \neq \sigma(X_s, D_s) = \sigma(X_s, p \cdot \mathrm{div}_{X_s}(x - \lambda^{1/p}))$$

$$\begin{tikzcd}
\dfrac{F_* \omega_X}{t} \arrow[r] \arrow[d, equal] & \dfrac{\omega_X}{t} \\
F_*(\omega_X / t^{p^e}) & F_*(\omega_X / t) \arrow[l] \arrow[u, dotted]
\end{tikzcd}$$

$F_* \omega_X \to \omega_X$, consider mod $t=0$

Relative Frobenius map (Radu-Andre map)

$$\begin{tikzcd}
X^e \arrow[dr, dotted, "F^e_{X/V}"] \arrow[drr, bend left, "F_X^e"] \arrow[ddr, bend right] & & \\
& X \times_V V^e \arrow[r] \arrow[d] & X \arrow[d, "F"] \\
& V^e \arrow[r, "F_V^e"] & V
\end{tikzcd}$$

If $s \in V$ is a point with perfect residue field, relative & absolute Frobenius coincide

We look at $\underset{\omega_{X^e}}{\overset{\|}{F_*^e \omega_X}} \xrightarrow{\star} \omega_{X \times_V V^e}$; this behaves well with respect to base change.

Define $\sigma_e(X/V) \cdot \omega_{X \times_V V^e}$ to be the image of $\star$.

Thm: $\exists e_0 > 0$ s.t. $\forall e \geq e_0$, $\sigma_e(X/V) \cdot \mathcal{O}_{V^e/m_{s^e}} = \sigma(X_s)$ for geometric $s \in V$.

Corollary: $\exists e_0 > 0$ s.t. $\forall e \geq e_0$, $\sigma(X \times_V V^e) \cdot \mathcal{O}_{X \times_V V^e / m_{s^e}} = \sigma(X_s)$, $\forall s \in V$.

note:
($\tau(X \times_V V^e) \cdot \mathcal{O}_{X/m_{s^e}} = \tau(X_s)$ $\forall s \in U$ open)

$\sigma_e(X/V) \otimes_{\mathcal{O}_{V^e}} \mathcal{O}_{V^{e+1}} \supseteq \sigma_{e+1}(X/V)$