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Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Bernard Teissier

Talk Title: On the local uniformization of Abhyankar valuations using toric maps

Date: 05 / 07 / 2013 **Time:** 2 : 00 **am / pm (circle one)**

List 6-12 key words for the talk: ______

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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On the local uniformization of Abhyankar valuations using toric maps

Bernard Teissier
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We will study valuations on an excellent equicharacteristic noetherian local domain R which are such that the inclusion $R \subset R_\nu$ in the valuation ring induces a trivial residue fields extension and the rational rank of the value group is equal to the dimension of R. Assuming that the residue field R/m is algebraically closed we will give an overview of their local uniformization and stress the fact that not only are they quasi-monomial, but they are toroidal in nature : the valuation ring is the limit of a nested system of birational toric maps between regular local rings.

On the local uniformization of Abhyankar valuations using toric maps

valuations $K^* \xrightarrow{\nu} \Phi$ totally ordered abelian group

$\nu(xy) = \nu(x) + \nu(y)$

$\nu(x+y) \geq \min(\nu(x), \nu(y))$

with equality if $\nu(x) \neq \nu(y)$

Zariski, 1960's: $x \in X$, $\mathcal{O}_{X,x} \subset K$, X/k, $\bar{k} = k$, ν of $K = k(X)$ trivial on $k \setminus \{0\}$. $\mathcal{O}_{X,x} \subset R_\nu \subset K$,

$m \cap \mathcal{O}_{X,x} = m_{X,x}$.

<u>Properness criterion</u>: X is proper$/k \Leftrightarrow$ every valuation has a center in X

$X' \longrightarrow X$ birational proper map, $\mathrm{Val}(K/k) = \varprojlim_{X' \to X} X'$.

$\nu \in \varprojlim_{X' \to X}$ $\quad \mathcal{O}_{X,x} \to \mathcal{O}_{X',x'} \to \dots$ $\quad R_\nu$ $\quad \mathcal{O}_{X,x} \subset \mathcal{O}_{X',x'} \subset R_\nu$.

ess. of finite type

If one can choose R' regular ($R \subset R' \subset R_\nu$), this is something like res. of singularities.

e.o.f.t.

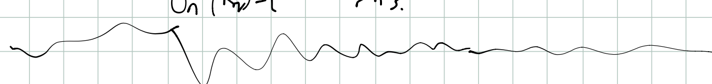
$$\begin{array}{ccccc} R & \subset & R' & \subset & R_\nu \\ \uparrow & & \uparrow & & \\ S & \subset & S' & & \end{array}$$

regular, birational $\leftarrow$ embedded local uniformization.

<u>Example</u>: R reg local ring, say $R = k[[x_1, \dots, x_k]]$, weights $w(x_i) > 0$ real values, $\nu(f)$ = order w/respect to weights.

$R/m \simeq R_\nu/m_\nu$. R_ν: filtration $\mathcal{P}_n(R_\nu) = \{x \in R_\nu \mid \nu(x) \geq n\}$.

$\mathcal{P}_n^+(R_\nu) = \{ \cdots \cdots > n\}$.



Let R be a Noeth. local ring, $R/m = k$, $\mathcal{P}_n(R)/\mathcal{P}_n^+(R)$, $\mathcal{P}_n^+(R) = \mathcal{P}_n^+(R_\nu) \cap R$, $gr_\nu R \subseteq gr_\nu R_\nu$.

$k[u_i \mid i \in I] \longrightarrow gr_\nu R$, $I = (u^k - \lambda_{mn} u^n)_{(m,n) \in \dots}$, $\lambda_{m,n} \in k^*$

$\Gamma = \nu(R \setminus \{0\}) \subseteq \Phi_+ \cup \{0\}$ semigroup of values. $\Gamma = \langle \gamma_1, \gamma_2, \dots \rangle$. $\dim \Gamma \leq \omega^{\dim R}$. $\quad \xi_i \in R \mapsto \bar{\xi}_i$.

$$\begin{array}{c} \mathcal{X} \\ \pi \downarrow \text{ faithfully flat} \\ \mathrm{Spec}\, k[t^{\Phi_+}] \end{array}$$

$\pi^{-1}(0) \simeq \mathrm{Spec}\, gr_\nu R$.

$\pi^{-1}(\eta) \simeq \mathrm{Spec}\, R$.

<u>Thm</u> (Piltant) k large (uncountable) $rk\, \Phi = rk_{\mathbb{Q}}(\Phi \otimes_{\mathbb{Z}} \mathbb{Q})$. Then $\dim gr_\nu R = tr_k k_\nu + rr\, \Phi \leq \dim R$.

Abhyankar $\longrightarrow$ $\dim gr_\nu R \leq \dim R$.

($= \Phi \subseteq \mathbb{Z}^{\oplus r}$)

In Φ_+, choose $\varphi_1, \dots, \varphi_n$ rationally independent. $\mathbb{N}^r \subset \mathbb{N}^r \subset \dots \subseteq \Phi_+ \cup \{0\}$.

Choose homogeneous elements $x_1^{(0)}, x_2^{(0)}, \dots, x_r^{(0)}$ rationally independent. $\exists$ sequence

$k[x_1^{(0)}, \dots, x_r^{(0)}] \to k[x_1^{(1)}, \dots, x_r^{(1)}] \to \dots \to k[x_1^{(h)}, \dots, x_r^{(h)}] \subset \dots \subset gr_\nu R_\nu \sim k_\nu[t^{\Phi_+}]$ (This is like a graded version of local uniformization)

Def: let $P = (u^{m^1} - \lambda_1 u^{n^1}, u^{m^2} - \lambda_2 u^{n^2}, \ldots, u^{m^r} - \lambda_r u^{n^r}) \subset k[[u_1, \ldots, u_N]]$. $m^i - n^i$ generate $\mathcal{L} \subset \mathbb{Z}^N \xrightarrow{b} \mathbb{Z}^r \to 0$

$\mathcal{L}$ is a direct factor of $\mathbb{Z}^N$.

$\operatorname{Spec} k[t^\Gamma] \subset \mathbb{A}^N(k) \,//\, \langle b(e_1), \ldots, b(e_N) \rangle = \Gamma$.

__Thm__ (Gonzalez Perez and —): $\operatorname{Spec} k[t^\Gamma]$ can be resolved by a toric map

$$\begin{array}{ccccc} \bigcup Z(\sigma) & \longrightarrow & Z(\Sigma) & \longrightarrow & \mathbb{A}^N_k \\ \cup & & \cup & & \\ X & \longrightarrow & \operatorname{Spec} k[t^\Gamma] & & \end{array}$$

$\Sigma = \{\alpha \in \check{\mathbb{R}}^N_+\}$, $\sigma = \langle a^1, \ldots, a^N \rangle$ basis of $\check{\mathbb{Z}}^N$

Weight on $k[[v_1, \ldots, v_N]]$, $w : \mathbb{Z}^N \to \mathbb{Z}^r$ (some ordering), $w(e_i) \in \mathbb{Q}_+$.

P prime binomial ideal as above.

$$\left.\begin{array}{l} F_1 = u^{m^1} - \lambda_1 u^{n^1} + \sum\limits_{w(p) > w(u^{m^1})} a_p^{(1)} u^p \\ \vdots \\ F_s = \text{(same, but with } i) \\ \\ \operatorname{in}_w(F_1, \ldots, F_s) = (\operatorname{in}_w F_1, \ldots, \operatorname{in}_w F_s) \end{array}\right]$$

This is called an <u>overweight</u> <u>deformation</u>

__Theorem__: Given an overweight deformation, $w(v_i) > 0$,

1. w determines a rational Abhyankar valuation on $k[[u_1, \ldots, u_N]]/(F_1, F_2, \ldots, F_s)$
2. Some of the toric maps resolving the binomials uniformize the valuations.

__Ex__: $(y^2 - x^3)^2 - x^5 y = 0$, $\mathcal{O} \subset k[[t]]$, $\operatorname{gr}_\nu \mathcal{O} = k[t^4, t^6, t^{13}] \subset k[t] = \operatorname{gr}_\nu R_\nu$. Then $P = (y^2 - x^3, z^2 - x^5 y)$ is a prime binomial ideal. $\exists$ overweight deformation $\begin{array}{l} y^2 - x^3 - z = 0 \\ z^2 - x^5 y = 0. \end{array}$

__Thm__ (Tevelev)

$$\begin{array}{ccccc} X' & \hookrightarrow & W & & \\ \downarrow & & \downarrow & & \\ X & \subset & \mathbb{P}^N & \hookrightarrow & \mathbb{P}^M \end{array}$$

, w/ projective coords say $w_0, \ldots, w_M$ s.t. $Z[\Sigma] \to \mathbb{P}^M$ is a ~

$R \subset R' \subset R_\nu$. $\exists R'$ such that $\Gamma' = \nu(R' \setminus \{0\})$ is fin. gen.

(R: excellent equichar. Noeth.)