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Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Bhargav Bhatt

Talk Title: A local Lefschetz theorem

Date: 05 / 07 / 2013 **Time:** 3 : 30 **am / pm (circle one)**

List 6-12 key words for the talk: ______

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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A local Lefschetz theorem

Bhargav Bhatt
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A classical theorem of Lefschetz asserts that non-trivial line bundles on a smooth projective variety of dimension >= 3 remain non-trivial upon restriction to an ample divisor. In SGA2, Grothendieck recast this result in purely local terms. Answering a question raised recently by Koll\' ar, we will explain how this local reformulation remains true under much milder hypotheses than those in SGA2. Our method uses a vanishing theorem in characteristic p, and formal geometry over certain very large (non-noetherian) schemes. This is joint work with Johan de Jong.

A local Lefschetz theorem.

(joint with J. de Jong)

I) Motivation:

<u>Thm (Lefschetz)</u>: $X/\mathbb{C}$ smooth proj. variety, $H \subset X$ ample. Then $H^i(X,\mathbb{Z}) \to H^i(H,\mathbb{Z})$ is $\begin{cases} \text{bijective if } i<\dim H \\ \text{injective if } i=\dim H. \end{cases}$

<u>Cor</u>: $X/\mathbb{C}$, H as above, H smooth. $\mathrm{Pic}(X) \to \mathrm{Pic}(H)$ is $\begin{cases} \text{bijective if } \dim X \geq 4 \\ \text{injective if } \dim X = 3 \end{cases}$

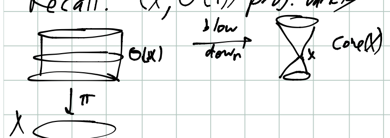
<u>pf of corollary</u>: Using the exponential sequence, we get:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \dfrac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})} & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \dfrac{H^1(H,\mathcal{O}_H)}{H^1(H,\mathbb{Z})} & \longrightarrow & \mathrm{Pic}(H) & \longrightarrow & H^2(H,\mathbb{Z}) \end{array} \quad \text{exact}$$

<u>Q</u>: Is there an algebraic proof, with fewer conditions?

II) <u>Local formulation</u> (from SGA2)

Recall: $(X, \mathcal{O}(1))$ proj. variety



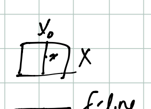
Principle: $\text{Alg Geom}/X \subseteq \text{Alg geom}/\mathrm{Cone}(X)\setminus\{x\}$

$H \subset X \rightsquigarrow f \in H^0(\mathrm{Cone}(X)\setminus\{x\}, \mathcal{O})$

$\dfrac{\mathrm{Pic}(X)}{\mathbb{Z}\mathcal{O}(1)} \xrightarrow{\sim} \mathrm{Pic}(\mathrm{Cone}(X)\setminus\{x\})$

<u>Notation</u>: $(A,\mathfrak{m})$ complete Noeth normal local ring of $\dim \geq 4$. $0 \neq f \in R$, $X = \mathrm{Spec}\,A \supset X_0 = \mathrm{Spec}\,A/(f)$

$\quad\quad V = X\setminus\{x\} \supseteq V_0 = X_0 \cap V$



<u>Thm G (SGA2)</u> Assume $\mathrm{depth}_x(A/(f)) \geq 3$. Then $\mathrm{Pic}(V) \xrightarrow{\sim} \mathrm{Pic}(V_0)$.

III) Kollár's conjecture:

<u>Observation</u>: Thm G is insufficient for applications to moduli theory.

<u>Conj. K</u>: $\mathrm{Pic}(V) \to \mathrm{Pic}(V_0)$ is injective if $\mathrm{depth}_x(A/(f)) \geq 2$, and injective up to p^∞-torsion if $\mathbb{F}_p \subset A$.

<u>Thm (B— & de Jong)</u>: Conj K is true as long as A contains a field.

Remarks:

1) The result is sharp.
2) $\exists$ a higher rank version for projective varieties in char p.
3) Kollár settled it when (A,m) is log canon. and m is not an lc center ($/\mathbb{C}$).
4) char $p \Rightarrow$ char 0 for our proof.
5) Only need excellent. (This is crucial for the reduction to char p.)

IV) Revisiting Thm G

More notation: $\left(\begin{array}{l} X_n = \operatorname{Spec}(A/(f^n)) \quad V_n = X_n \cap V \\ V_0 \subset V_1 \subset V_2 \subset \cdots \end{array} \right.$, $\hat{V}$ = formal completion of V along V_0 "=" $\operatorname{colim}_n V_n$

Idea: Use $\operatorname{Pic} V \xrightarrow{a} \operatorname{Pic} \hat{V} \xrightarrow{b} \operatorname{Pic} V_0$, $\operatorname{Pic} \hat{V} \cong \lim_n \operatorname{Pic} V_n \longrightarrow \operatorname{Pic} V_0$

Show a, b are injective

Lemma: $\mathcal{E} \in \operatorname{Vect}(V)$, then $\widehat{H^0(V,\mathcal{E})} = H^0(\hat{V},\hat{\mathcal{E}})$.

Derived formal functions theorem: $\widehat{R\Gamma(V,\mathcal{E})} = R\Gamma(\hat{V},\hat{\mathcal{E}})$

(where for a complex K, $\hat{K} :=$

$\Rightarrow \widehat{H^0(V,\mathcal{E})} \longrightarrow H^0(\hat{V},\hat{\mathcal{E}}) \longrightarrow T_f H^1(V,\mathcal{E})$ ("f-adic Tate module"), $T_f H^1(V,\mathcal{E}) = \lim_n H^1(V,\mathcal{E})[f^n]$. So it's enough to show $T_f H^1(V,\mathcal{E}) = 0$.

$\Leftarrow H^1(V,\mathcal{E})[f^\omega]$ is bounded

$\Leftrightarrow H^1(V,\mathcal{O}_V)[f^\infty]$ is bounded.

$H^1(V,\mathcal{O}_V) = 0$ by normality.

Cor: $\operatorname{Vect}(V) \to \operatorname{Vect}(\hat{V})$ is fully faithful $\Rightarrow \operatorname{Pic}(V) \to \operatorname{Pic}(\hat{V})$ is injective.

Lemma 2: $\operatorname{Pic}(V_{n+1}) \to \operatorname{Pic}(V_n)$ is injective if $H^1(V_0, \mathcal{O}_{V_0}) = 0$.

Pf: $1 + f^n \mathcal{O}_{V_{n+1}} \longrightarrow \mathcal{O}^*_{V_{n+1}} \longrightarrow \mathcal{O}^*_{V_n}$, $1 + f^n \mathcal{O}_{V_{n+1}} \cong \mathcal{O}_{V_0}$

~~Long exact seq.~~ does the rest.

Pf of Thm C. $H^1(V_0, \mathcal{O}_{V_0}) = 0$ by assumption. $\operatorname{Pic}(V) \xrightarrow{\text{inj}} \operatorname{Pic}(\hat{V}) \xrightarrow{\text{inj}} \operatorname{Pic}(V_n) \xrightarrow{\text{inj}} \operatorname{Pic}(V_0)$.

V) Main thm:

Assume A has char $p > 0$. Goal is to show $\operatorname{Pic}(V) \longrightarrow \operatorname{Pic}(V_0)$ inj. Problem: $H^1(V_0, \mathcal{O}_{V_0}) \neq 0$

Def: $\bar{A}$ = abs. int. closure of A ($= A^+$).

Thm (Hochster–Huneke): $\bar{A}$ is Cohen–Macaulay. In particular $H^i(\bar{V}, \mathcal{O}_{\bar{V}}) = 0$, $i < \dim V$

$\bar{X} = \operatorname{Spec} \bar{A}$, $\bar{V} = \bar{X} \setminus \{x\}$. $\qquad H^i(\bar{V}_0, \mathcal{O}_{\bar{V}_0}) = 0$, $i < \dim V - 1$

Check: previous argument for Thm G goes through over $\overline{A}$.

Now use
$$\begin{tikzcd} \mathrm{Pic}(V) \arrow[r, "f"] \arrow[d, "g"] & \mathrm{Pic}(\overline{V}) \arrow[d] \\ \mathrm{Pic}(V_0) \arrow[r] & \mathrm{Pic}(\overline{V}_0) \end{tikzcd}$$
. We have $\ker g \subset \ker f =$ torsion.

$\Rightarrow \ker g$ is torsion. $\square$

Thm (– & d.J., Langer): X/k normal proj. variety of char $p>0$, $\dim \geq 3$, $\mathcal{E} \in \mathrm{Vect}(X)$ s.t. $\mathcal{E}/H \cong \mathcal{O}_H^{\oplus r}$ for $H \subset X$ ample

$\Rightarrow (\mathrm{Frob}_X^e)^* \mathcal{E} \simeq \mathcal{O}_X^{\oplus r}$.

pf: Use $\overline{X}$.