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Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Tommaso de Fernex

Talk Title: The Nash problem on families of arcs

Date: 05 / 08 / 2013 **Time:** 9:00 am / pm (circle one)

List 6-12 key words for the talk: ______

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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The Nash problem on families of arcs

Tommaso de Fernex
University of Utah

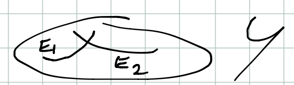
Hironaka's theorem on resolution of singularities allows to study the geometry of a singular variety (over a field of characteristic zero) by looking at a smooth birational model. A more intrinsic approach to study singularities was proposed by Nash. The idea is to look at the space of arcs passing through the singular points. This space decomposes into finitely many irreducible families, and carries some of the information encoded in a resolution. The Nash problem gives a precise formulation of how such families of arcs should relate to resolutions of singularities. In this talk I will give an overview of the history and solution of the problem.

The Nash problem on families of arcs

example: $X = (xy - z^3 = 0) \subset \mathbb{A}^3$. Look at arcs through the origin: $\gamma(t) = \begin{cases} x = a_1 t + a_2 t^2 + \cdots \\ y = b_1 t + \cdots \\ z = c_1 t + \cdots \end{cases}$

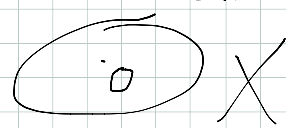
w/ the condition that $x(t)y(t) = z(t)^3$

$\Rightarrow a_1 b_1 = 0 \rightsquigarrow$ two families $\begin{array}{l} C_1 = \{a_1 = 0\} \\ C_2 = \{b_1 = 0\}. \end{array}$



$\downarrow f$ $\quad \begin{array}{l} x = u^2 v \\ y = u v^2 \\ z = uv \end{array}$ $\quad$ where $E_1 = (u=0)$, $E_2 = (v=0)$

$\gamma_1 = \begin{cases} x = t\bar{x} \\ y = t\bar{y} \\ z = t\bar{z}. \end{cases}$



lifts to $\tilde{\gamma}_1 = \begin{cases} u = \dfrac{t^2 \bar{x}}{t\bar{z}} = t \cdot \text{unit} \\ v = \dfrac{t\bar{y}}{t\bar{z}} = \text{unit}. \end{cases}$

Correspondence:
$C_i \longleftrightarrow E_i$

Nash problem: $\gamma : \operatorname{Spec} \mathbb{C}[[t]] \to X$

$$\begin{array}{l} \text{arc space } X_\infty \supseteq \pi^{-1}(\operatorname{Sing} X) = C_1 \cup \cdots \cup C_s \quad \text{irred. components} \\ \qquad \downarrow \pi \\ \text{resolution } Y \xrightarrow{f} X \supset \operatorname{Sing} X \\ \qquad\qquad \mathbb{C}\text{-variety} \end{array}$$

1. $f^{-1}(\operatorname{Sing} X) = E_1 \cup \cdots \cup E_r$ irreducible components

valuations: $E_j \rightsquigarrow \operatorname{val}_{E_j}$ $\qquad E_j \subset Bl_{E_j} Y$

$C_i \rightsquigarrow \operatorname{val}_{C_i}$

generic $\ni \alpha_i : \operatorname{Spec} K[[t]] \to X$, where $K \mid \mathbb{C}$ is a field ext.

$$\mathbb{C}(X)^* \xrightarrow{\alpha^*} K((t))^* \xrightarrow{\operatorname{ord}_t} \mathbb{Z}$$

$$\begin{array}{ccccc} \pi_Y^{-1}(E_j) \subset & Y_\infty & \xrightarrow{f_\infty} & X_\infty \supseteq & f_\infty(\pi_Y^{-1} E_j) = N_{E_j} \\ & \downarrow \pi_Y & & \downarrow \pi_X & \\ E_j \subset & Y & \xrightarrow{f} & X & \end{array}$$

maybe redundant!

$$\pi_X^{-1}(\operatorname{Sing} X) = \bigcup C_i = \bigcup N_{E_j} = \bigcup \overline{N}_{E_j}.$$

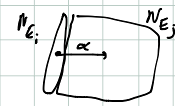
In fact, $\forall i \exists j$ s.t.
$C_i = \overline{N}_{E_j}$ & $\operatorname{val}_{C_i} = \operatorname{val}_{E_j}$.

This determines the Nash map: $\{C_1, \ldots, C_s\} \longrightarrow \{E_1, \ldots, E_r\}$
$C_i \longmapsto E_j$

Problem: characterize the image!

Dim 2 case: take f minimal resolution.

Thm (de Bobadilla, Pereira): In dim 2, the Nash map is bijective.

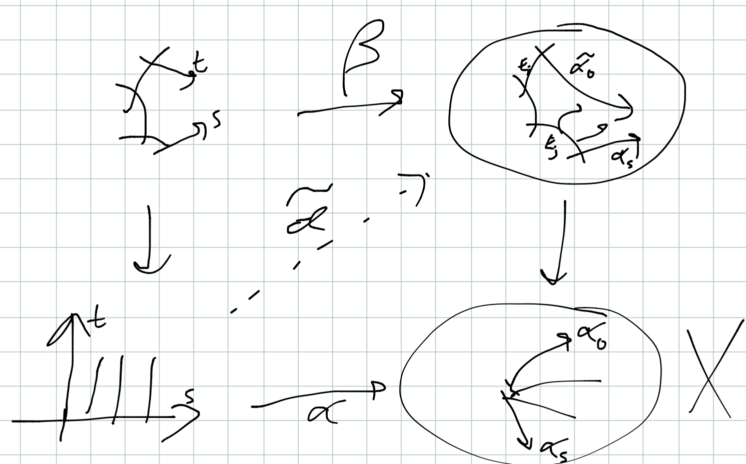


pf by contradiction: Assume $\exists\, i \neq j$ s.t. $N_{E_i} \subset \overline{N_{E_j}}$.

Lejeune-Jalabert, Reguera: $\exists$ arc $\alpha : \mathrm{Spec}\, \mathbb{C}[[s]] \to X_\infty$, $\alpha(0) \in N_{E_i}$, $\alpha(\eta) \in N_{E_j}$.

(Example: $N = \mathrm{Spec}\, \dfrac{\mathbb{C}[[x_1, x_2, x_3, \dots]]}{(x_1 - x_2^2, x_1 - x_3^3, x_1 - x_4^4, \dots)}$. Then $\nexists$ nontrivial arcs through $0 \in N$.)

Think of $\alpha : \mathrm{Spec}\, \mathbb{C}[[s,t]] \to X$, $\alpha(s,t) = \alpha_s(t)$.

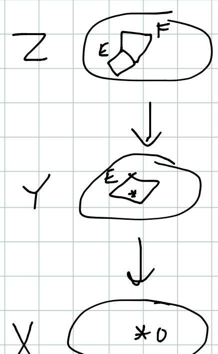


Dim ≥ 3: $\nexists$ minimal model

Def: A divisor E over X is essential if $\forall$ resol. $f : Y \to X$, $\mathrm{center}_Y(\mathrm{val}_E) =$ irred comp. of $f^{-1}\mathrm{Sing}\,X$

Nash problem: $\{$irred comps of $\pi_X^{-1}\mathrm{Sing}\,X\} \overset{?}{=} \{$Essential divisors over $X\}$

Ishii-Kollár: counterexamples in dim ≥ 4.



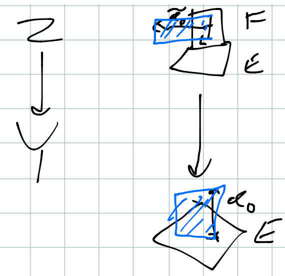
2-blowup resol.

1. F is not birationally ruled
2. F is covered by lines L s.t. $H^1(L, N_{L/F}) = 0$.

1&2 $\Rightarrow \dim X \geq 4$.

$1 \Rightarrow$ F is essential.

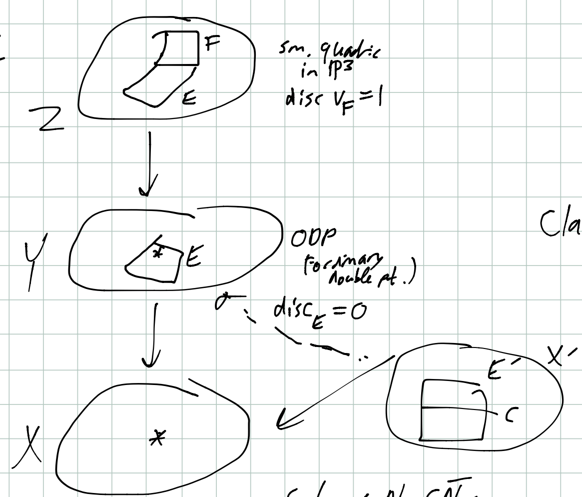
$2 \Rightarrow N_F \subset \overline{N_E}$



Contribution of (–): counterexamples in dim3

and Nash problem depends on topology (Zariski vs. analytic)

<u>Ex 1:</u>



Claim: F is essential

Claim: $N_F \subset \overline{N_E}$.