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Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Kevin Tucker

Talk Title: F-signature and relative Hilbert-Kunz multiplicity

Date: 05 / 08 / 2013 **Time:** 10:30 am / pm (circle one)

List 6-12 key words for the talk: ______

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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F-Signature and Relative Hilbert-Kunz Multiplicity

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The F-signature and Hilbert-Kunz multiplicity of a local ring are among a number of numerical invariants defined via Frobenius in positive characteristic. After introducing these so-called F-invariants, we will sketch a partial answer to a question of Watanabe and Yoshida by showing that the F-signature and relative Hilbert-Kunz multiplicity (for cyclic modules) coincide. The method of proof also suggests a number of generalizations of both invariants that will be described as time allows.

F-signature and relative Hilbert-Kunz multiplicity

Setup: $(R, \mathfrak{m}, k)$ dim d Complete local domain, char $p>0$, $k = k^p$

$$F: R \longrightarrow R$$
$$r \longmapsto r^p$$

M R-mod $\rightsquigarrow$ $F_*M = M$ viewed as an R-module by restriction of scalars

Example: $R = \mathbb{F}_p[[x_1, \dots, x_d]]$, $F_*^e(R) \cong R^{1/p^e} = \mathbb{F}_p[[x_1^{1/p^e}, \dots, x_d^{1/p^e}]]$ free R-module

with basis $\{x_1^{a_1} \cdots x_d^{a_d}\}$, $0 \le a_i < 1$, $a_i \in \frac{1}{p^e}\mathbb{Z}$, so rank $= p^{ed}$

Thm (Kunz): R regular $\iff$ F is flat $\iff$ $F_*^e R$ free of rank p^{ed}, some/all $e>0$.

Def: $F_*^e R = R^{\oplus a_e} \oplus M_e$, $R \nmid M_e$. Then a_e is called the e^{th} F-splitting number of R.

(in our example, $a_e = p^{ed}$)

Example: $R = \dfrac{\mathbb{F}_p[[x,y,z]]}{(x^3+y^3+z^3)}$. Then $a_e = \begin{cases} 1, & p \equiv 1 \bmod 3 \\ 0, & p \equiv 2 \bmod 3 \end{cases}$.

Def: R is F-split if $R \to F_*^e R$ splits (as R-modules) for some/all $e>0$.

($\iff \exists\ \varphi \in \mathrm{Hom}_R(F_*^e R, R)$ surjective for some/all R $\iff$ $a_e > 0$ for some/all $e>0$)

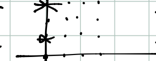
Def: $P := \{ r \in R \mid \begin{array}{l} R \longrightarrow F_*^e R \\ 1 \longmapsto F_*^e r \end{array}$ is not split for any $e>0\}$ is the F-splitting prime of R. $n := \dim R/P$ is the F-splitting dimension of R.

Thm: $r_F(R) = \lim_{e\to\infty} a_e / p^{en}$, where n is the F-splitting dimension of R.

(called the F-splitting ratio)

It exists (T) and if R is F-split, it is positive (Blickle-Schwede-T.)

Example: $p \ne 2$, $R = \mathbb{F}_p[[a,b,c]]/(b^2 - a^2 c)$, $a_e = \frac{p^e+1}{2}$, $P = \langle a, b\rangle$, $d=2$, $n=1$.

Note that R is a seminormal monoid ring. $R \subseteq \mathbb{F}_p[[$  $]] = \mathbb{F}_p[[x, xy, y^2]]$.

Let $M = \mathbb{N}^d \setminus \{(a_1, \dots, a_d) \mid$ one of $a_1, \dots, a_k$ is odd$\}$. Let $R = \mathbb{F}_p[[M]]$. Then $r_F(R) = \frac{1}{2^k}$ and $n = k$.

(joint w/ Hering)

Def: The F-signature of R is $s(R) = \lim_{e\to\infty} \dfrac{a_e}{p^{de}}$

note: R is strongly F-regular $\iff P = 0$ (i.e. $n=d$) $\iff s(R) > 0$ (Aberbach-Leuschke)

R is regular $\iff s(R) = 1$ (Yao)

Ex (Huneke-Leuschke, using a result of Watanabe-Yoshida)

ADE singularities in dim 2.

$s(A_n) = 1/(n+1)$

$s(D_n) = 1/4(n-2)$

$s(E_6) = 1/24, \ s(E_7) = 1/48, \ s(E_8) = 1/120$

In general, $s(R) = \frac{1}{|G|}$, where G defines the quotient singularity. Similar formulas hold for all tame quotients. (Yao, T.)

Conjecture [Monsky]: $R = \mathbb{F}_2[[x,y,z,u,v]]/(uv + xyz + x^3 + y^3)$. Then $s(R) \overset{?}{=} \frac{2}{3} - \frac{5}{14\sqrt{7}}$.

Thm/Def (Monsky). Let $I \subset R$ be $\mathfrak{m}$-primary. The Hilbert-Kunz mult. of I is

$$e_{HK}(I) = \lim_{e\to\infty} \frac{\ell(R/I^{[p^e]})}{p^{ed}}, \text{ where } I^{[p^e]} := \langle i^{p^e} \mid i \in I \rangle. \text{ This exists.}$$

Conj (Monsky): $e_{HK}(\mathfrak{m}) = \frac{4}{3} + \frac{5}{14\sqrt{7}}$.

Example [Han-Monsky]: $p \neq 2$, $R = \mathbb{F}_p[[x,y,z,w]]/\langle x^2+y^2+z^2+w^2 \rangle$. $\ell(R/\mathfrak{m}^{[p^e]}) = \frac{4}{3}p^{3e} - \frac{1}{3}p^e$

Thus $e_{HK}(\mathfrak{m}) = 4/3$.

Ex (Shideler): $p \equiv 1 \pmod 3$, $n = \frac{p-1}{3}$, $R_p = \mathbb{F}_p[[x,y,z,w]]/\langle x^3+y^3+z^3+w^3 \rangle$

$$a_e = \underbrace{\frac{27n^2+27n+6}{216n^2+216n+64}}_{s(R_p)} p^{3e} + \frac{192n^2+192n+58}{216n^2+216n+64}(n+1)^e$$

Note: $\lim_{p\to\infty} s(R_p) = 1/8$.

main theorem!

Conj (Watanabe-Yoshida) / Thm (+): $s(R) = \inf_{I \subsetneq J} (e_{HK}(I) - e_{HK}(J)) =: \text{"}e^{rel}_{HK}(I \subseteq J \subseteq R)\text{"}$

Thm (Hochster-Huneke)

$e^{rel}_{HK}(I \subseteq J \subseteq R) = 0 \iff J \subseteq \overset{\text{tight closure}}{I^*} \iff \bigcap_{e>0} (I^{[p^e]} : J^{[p^e]}) \neq 0$.

Def: R is weakly F-regular if $I^* = I$ $\forall$ ideals I. ($\overset{\text{by the thm}}{\iff} e_{HK}(I \subsetneq J \subseteq R) > 0 \ \forall I \subsetneq J$)

Question: Is the inf a min?

Note: Yes $\Rightarrow$ (Weak F-reg = strong F-reg.)

Sketch pf of result

Main tool: improved limit existence thms. w/ better "uniformity".

Fix $0 \neq \sigma \in Hom_R(R, F_* R)$ and $0 \neq \psi \in Hom_R(F_* R, R)$.

let $\{I_e\}$ be a seq. of ideals, m-primary, s.t. $m^{[p^e]} \subseteq I_e$, such that
① $\sigma(I_e) \subseteq F_* I_{e+1}$
and ② $\psi(F_* I_{e+1}) \subseteq I_e$.

Thm (T): $\lim_{e\to\infty} \frac{\ell(R/I_e)}{p^{ed}} = \gamma$ exists and $\ell(R/I_e) = \gamma p^{ed} + O(p^{e(d-1)})$

same unit const $\forall$ such seqs $\{I_e\}$.

Examples: $\cdot\ I_e = m^{[p^e]} \rightsquigarrow e_{HK}$

$\cdot\ I_e = \{r \in R \mid \underset{1 \mapsto F_*^e r}{R \to F_*^e R} \text{ not split}\} \rightsquigarrow s(R)$.

$\cdot$ any two ideals $I \subseteq J$, $I \subseteq J$, $\ell(J/I) = 1$. $I_e = (I^{[p^e]} : J^{[p^e]}) \rightsquigarrow e_{HK}^{rel}(I \subseteq J \subseteq R)$.

Now let $\{J_t\}$ be an approx Gorenstein sequence. Let u_t generate the socle of R/J_t

$\xRightarrow{\text{YAO}} I_e = (J_t^{[p^e]} : u_t^{[p^e]})\quad \forall t >> 0$

$$s(R) = \lim_{e\to\infty} \lim_{t\to\infty} \frac{\ell(R/(J_t^{[p^e]} : u_t^{[p^e]}))}{p^{ed}} = \left(\text{interchange limits!}\right) = \lim_{t\to\infty} e_{HK}(J_t \subseteq (J_t, u_t) \subseteq R) \quad \square$$

Why can you interchange the limits?

because $\left|\gamma - \frac{\ell(R/I_e)}{p^{ed}}\right| \leq \frac{C}{p^e}$, C a uniform constant.