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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Vijaylaxmi Trivedi

Talk Title: Some computations of Hilbert-Kunz functions

Date: 05 / 08 / 2013 **Time:** 11 : 30 am / pm (circle one)

List 6-12 key words for the talk: ______

Please summarize the lecture in 5 or fewer sentences: The speaker proposes a limit relating Hilbert-Kunz and ordinary multiplicities associated to successive Veronese embeddings of a projective variety with respect to line bundles. She computes the limit (hence showing that it exists) in some special cases.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
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  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Some computations of Hilbert-Kunz functions

R comm. Noeth. ring of char $p>0$. $I \subseteq R$ ideal s.t. $\ell(R/I)<\infty$.
R is standard graded and I is a graded ideal.

$q=p^n$
$HK(R,I)(q)=\ell(R/I^{[q]})$, $I^{[q]}=(x^q \mid x\in I)$

Monsky: $e_{HK}(I)$ exists.
That is, $HK(R,I)(q)=e_{HK}(R,I)q^d+O(q^{d-1})$.

Note: $\frac{e(R)}{d!}\le e_{HK}(R)\le e(R)$.

$\dim R=2$: $\frac{e(R)}{2}\cdot\frac{\operatorname{embdim}(R)}{\operatorname{embdim}(R)-1}\le e_{HK}(R)$.

Watanabe-Yoshida: R Noeth local ring $I\subseteq R$ $\frac{e_{HK}(I^n)}{d!}\le e_{HK}(I^n)\le\binom{n+d-1}{d}e(I^n)$ and $e_{HK}(I^n)=\frac{e(I^n)}{d!}+O(n^{d-1})$

X: projective variety, $\mathcal{L}$ = line bundle of degree n_0, $R=\bigoplus_{m\ge 0}H^0(X,\mathcal{L}^m)$, n^{th} Veronese embedding $X^n \hookrightarrow \|$ (?)
have coordinate ring $R^n := \bigoplus_{m\ge 0}H^0(X,\mathcal{L}^{mn})$.

Q: $\lim_{n\to\infty}\frac{e_{HK}(R^n)-e(R^n)/(d+1)!}{n^{d-1}}=?$ (Does the lim exist? If so, what is it?)

Ex: X = elliptic curve, $\mathcal{L}_0$ a line bundle of degree n_0. $e_{HK}(R^n)=\frac{(nn_0)^2}{2(nn_0-1)}$, $e(I^n)=nn_0$. $\lim_{n\to\infty} e_{HK}(R^n)-\frac{e(R^n)}{2}=\frac{1}{2}$.

Ex: $X=G/B$, $G=SL_{r+1}(k)$, B = upper triangular matrices, T = matrices w/ diagonal action, $\mathcal{L}_0=\mathcal{L}(kP)$, P = sum of fundamental weights.
$k=1$, $\bigoplus_{m\ge 0}H^0(X,\mathcal{L}(mp))\cong k[A_1,\dots,A_r \mid A_e\in\Lambda_e]\subseteq k[X_{ij}]$, where $[X_{ij}]$ is $r\times r+1$ and $\Lambda_e=\{e\times e \text{ minor of first } e \text{ rows}\}$

$R^n\sim n^{th}$ Veronese embedding of $(X,\mathcal{L}(kP))$, $e_{HK}(R^n)=\frac{(nk+1)^d}{d+1}$, $e(R^n)=(nk)^d\cdot d!$, so $\lim_{n\to\infty}\frac{e_{HK}(R^n)-\frac{e(R^n)}{(d+1)!}}{n^{d-1}}=\frac{(k+1)^d-k^d}{d+1}$.

Ex: X = Hirzebruch surface $=\overline{H}_a$, $a\ge 2$, $\mathcal{L}$ line bdl. on X, $R=\bigoplus_{m\ge 0}H^0(X,\mathcal{L}^{\otimes m})$, $\ell(R/I^{[q]})=\sum_{m\ge 0}\ell((R/I^{[q]})_m)$

$R_1^{(q)}\to R_{m+q}$, $\operatorname{Coker}=\ell((R/I^{[q]})_{m+q})$, $H^0(X,\mathcal{L})^{(q)}\otimes H^0(X,\mathcal{L}^m)\longrightarrow H^0(X,\mathcal{L}^{m+q})$

$$\begin{array}{ccc} \| & \cong & \cong \\ \operatorname{Coker}\alpha_m & H^0(X,\mathcal{L})\otimes H^0(X,F_*\mathcal{L}^m) \xrightarrow{\alpha_m} & H^0(X,F_*\mathcal{L}^{m+q}) \end{array}$$

(She draws a picture)

Ex: X a toric variety, $\Sigma\subseteq N^d=N^2$, $M=\operatorname{Hom}(N,\mathbb{Z})$.

$$0\to M\to \operatorname{Div}(T)\longrightarrow \operatorname{Pic}(X)\to 0$$
$$x\longmapsto \sum \chi(v_i)\cdot D_i$$

$\frac{1}{q}$: $\frac{M}{qM}\longrightarrow \operatorname{Pic}(X)$
$\chi\longmapsto\left\lfloor\frac{D_\chi}{q}\right\rfloor$.

Lasón & Michałek: Let $L=\sigma(D)$ line bdl on divisor D on smooth toric variety. Then $F_*\sigma(D)=\bigoplus_{\chi\in M/qM}\sigma\left(\left\lfloor\frac{D+D_\chi}{q}\right\rfloor\right)$
$=\bigoplus_{\chi\in M/qM}L_\chi$

We need to know which L_χ occur and with what multiplicities.

[In the example she drew on the board]: Pic(X) is free on two generators D_1 & D_4.

$X = F_a = \mathbb{P}(\mathcal{O}(a)\oplus\mathcal{O}(1))$, $\mathcal{E} = \mathcal{O}(a)\oplus\mathcal{O}(1)$. $D_1 = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$, $D_4 \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$

$\downarrow\pi$

$\mathbb{P}^1$

$a_1D_1 + a_2D_2 + a_3D_3 + a_4D_4$

$\sim (a_1 - a a_2 + a_3)D_1 + (a_2 + a_4)D_4$.

$\mathcal{L} = cD_1 + dD_4$ is globally gen'd $\Leftrightarrow c \geq 0\ d \geq 0 \Leftarrow$ ample.

(Work in progress): $\mathcal{L} = cD_1 + dD_4$, $c = d$, $a \geq 2$, $F_*\mathcal{L}^m = \bigoplus_{\alpha,\beta} \mathcal{O}(\alpha D_1 + \beta D_4)^{|A_{\alpha,\beta}|}$

$|A_{\alpha,\beta}| = \{x \in M/qM \mid \mathcal{O}(\lfloor \frac{mD + D_x}{q} \rfloor) \sim \alpha D_1 + \beta D_4\}$

$m < q/d$, $\{-1, 0, 1, \ldots, a\} \ni \alpha$, $\beta \in \{0, -1\}$, $H^0(cD_1 + dD_4) \otimes H^0(\alpha D_1 + \beta D_4) \longrightarrow H^0(\quad \otimes \quad)$

$(\mathrm{Coker}(\phi_m))_{(\alpha,\beta)} = d \cdot \lfloor (d + \alpha + 1) + \frac{d-1}{2} a \rfloor$, $\alpha \geq 0, \beta = -1$, $\alpha = -1, \beta = 0$

(now $\exists$ another picture)

$\sum_{m \geq 0} \mathrm{coker}\, \phi_m = q^3(\frac{1}{3} + d) + q^2(\quad) + q(P(d, \varepsilon_1, \varepsilon_3)) + q^0(P(d, \varepsilon_1, \varepsilon_2, \varepsilon_3))$, $\varepsilon_\ell = \lceil q/\ell d \rceil - \frac{q}{\ell d}$

Then $\lim_{d\to\infty} \frac{e_{HK}(R^d) - \frac{e(R^d)}{a}}{d} = \frac{\frac{1}{3} + d}{d} = \frac{1}{3}$