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Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Shunsuke Takagi

Talk Title: Globally F-regular and Frobenius split surfaces

Date: 05 / 09 / 2013 **Time:** 9 : 00 am / pm (circle one)

List 6-12 key words for the talk: ______

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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Globally F-regular and Frobenius split surfaces.

Shunsuke Takagi
University of Tokyo

Frobenius split and Globally F-regular varieties are classes of projective varieties over a field of positive characteristic, defined in terms of Frobenius splitting. I will explain some properties of Frobenius split and globally F-regular surfaces.

This is joint work with Yoshinori Gongyo.

Globally F-regular and Frobenius split surfaces

— joint w/ Y. Gongyo

Def: Let X be a normal proj. variety$/k=\bar{k}$, char p. Let $F: X \to X$ be the abs. Frob. map.

(1) We say X is *globally F-split* if $\mathcal{O}_X \to F_* \mathcal{O}_X$ splits.

(2) We say X is *globally F-regular* if $\forall D \geq 0$ Cartier divisor on X, $\exists e > 0$ s.t. the composition $\mathcal{O}_X \to F^e_* \mathcal{O}_X \to F^e_* \mathcal{O}_X(D)$ splits.

Note: gl. F-reg $\Rightarrow$ gl. F-split.

Ex: Let C be a smooth projective curve in char $p > 0$.

C is gl. F-reg $\Leftrightarrow C \cong \mathbb{P}^1$

C is gl. F-split $\Leftrightarrow C \cong \mathbb{P}^1$ or ordinary elliptic curve

Thm (Schwede-Smith): · If X is globally F-regular, X is log Fano. (i.e. $\exists \Delta \geq 0$ s.t. (X, Δ) is klt and $-(K_X+\Delta)$ is ample.)

· If X is globally F-split, then X is log CY (Calabi-Yau). (i.e. $\exists \Delta \geq 0$ s.t. (X,Δ) is l.c. and $K_X + \Delta \sim_{\mathbb{Q}} 0$.)

Note: The converse fails for both statements.

Ex: $X = (x^3+y^3+z^3+w^3=0) \subset \mathbb{P}^3$, $p=2$. Then X is Fano but not globally F-split.

Def: X normal proj. variety$/k=\bar{k}$, char $k=0$.

(1) X is *of globally F-regular type* if its modulo p reduction X_p is gl. F-reg. $\forall p \gg 0$.

(2) X is *of dense globally F-split type* if its modulo p reduction X_p is gl. F-split for infinitely many p.

Ex: E elliptic curve in char 0. $\exists$ infinitely many p s.t. E_p is ordinary. (Serre)
Hence, E is of dense gl. F-split type.

Conj. (Schwede-Smith):

(1) X is of gl. F-regular type $\Leftrightarrow$ X is log Fano.

(2) X is of dense gl. F-split type $\Leftrightarrow$ X is log CY.

Rmks: ($\Rightarrow$) does not follow from the previous thm.

· ($\Leftarrow$) in (1) is true (Schwede-Smith)

· ($\Leftarrow$) in (2) is true if the weak ordinarity conjecture holds.

We focus on ($\Rightarrow$).

($\Rightarrow$) is true if X is a MDS (Gongyo-Okawa-Sannai-T.)

(MDS means Mori dream space: Let $D_1, \dots, D_r$ be a basis for $Cl(X)$. $Cox(X) = \bigoplus_{(m_1,\dots,m_r)\in \mathbb{Z}^r} H^0(X, m_1D_1+\dots+m_rD_r)$. X is MDS if $Cox(X)$ is f.g.)

($\Rightarrow$) in (1) is true if $\dim X = 2$ (Okawa)

Thm (Gongyo-T.): If X is a globally F-regular surface, then X is log Fano.
If X is a dense globally F-regular type surface, then X is log CY.

__pf. of gl. F-split case:__

Taking minimal resolution, WMA X is smooth. X_p is globally F-split for infinitely many p. So $-K_{X_p}$ is pseudo-effective for ∞ many p, whence $-K_X$ is pseudo-effective.

$\exists$ Zariski decomposition $-K_X = P + N$.

__Lemma__: Then $-K_{X_p} = P_p + N_p$ is a Zariski decomp. for ∞ many p.

(X_p, N_p) is globally F-split

__Def__: A pair (Y, M) is __globally F-split__ if $\exists e > 0$, s.t. the composition $\mathcal{O}_Y \to F^e_* \mathcal{O}_Y \hookrightarrow F^e_* \mathcal{O}_Y(\lceil p^e - 1\rceil M)$ splits.

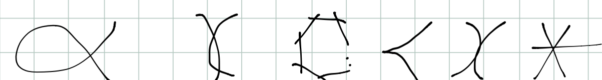
Then by Hara-Watanabe, (X, N) is log canonical.

__ETS__: P is semiample.
$(B \in |nP|$ general $(n \gg 0) \Rightarrow (X, \frac{1}{n}B + N)$ is lc. So $K_X + \frac{1}{n}B + N \sim_{\mathbb{Q}} K_X + P + N \sim 0)$

We can reduce to the following situation: X smooth rational surface, $-K_X$ is nef but not semiample
$0 \neq H^0(X, -K_X)$ — i.e. $\exists D \geq 0$ s.t. $D \sim -K_X$.

$D_p \sim -K_{X_p}$ is semiample. $\pi : X_p \to \mathbb{P}^1$ elliptic fibration, minimal.
Then $\exists x \in \mathbb{P}^1$ s.t. $D_p = \frac{1}{m}\pi^* x$, where $m =$ multiplicity of $\pi^* x$.

__ETS__: (X, D) is lc.
Use classification of singular fibers of elliptic fibrations. (Deligne-Mumford?)



I_1 I_2 I_b II III IV

Suppose (X, D) is not lc. Then D_p is not normal crossing (i.e. $D \neq I_1, I_2$, or I_b, $b \geq 3$).
Then $H^0(X_p, (1-p)K_{X_p}) = H^0(X_p, (1-p)D_p) = 1$. But $H^0(X_p, (1-p)K_{X_p}) \cong \mathrm{Hom}(F_* \mathcal{O}_{X_p}, \mathcal{O}_{X_p})$ (Groth. duality)

$(p-1)D_p \longleftrightarrow \exists \varphi$ splitting map.

Then (X_p, D_p) is gl. F-split. Thus (X, D) is lc, contradiction.