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Speaker's Name: Masayuki Kawakita

Talk Title: The index of a threefold canonical singularity

Date: 05 / 09 / 2013 **Time:** 10 : 30 am / pm (circle one)

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Please summarize the lecture in 5 or fewer sentences: (see abstract)

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The index of a threefold canonical singularity

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We will discuss the question raised by Shokurov: can one bound the index of a Q-Gorenstein singularity in terms of the discrepancies over it? We will give an answer for 3-fold canonical singularities.

The index of a threefold canonical singularity

$P \in X/\mathbb{C}$ normal, $\mathbb{Q}$-Gor (i.e. rK_X Cartier, index r_P = the smallest $r>0$)

Shokurov: Can bound r_P in terms of discrepancies of divisors over $P \in X$

$E \subset Y \xrightarrow{f} X$ (prime div., proper birat.) discrep. $a_E(X) :=$ coeff of E in $K_Y - f^*K_X$, center $c_X(E) := f(E)$

X term. $\overset{def}{\iff} a_E(X) > 0$ $\forall E$ excep $\overset{\dim 2}{\iff}$ sm Gor $\}$ $r_P = 1$

cano	≥ 0
lt	> -1
lc	≥ -1

Want to bound r_P by fixing the min discrepancy $md_P(X) := \min_{c_X(E)=P} a_E(X) \in \{-\infty\} \cup [0,\infty)$ $\Leftarrow$ $P\in X$ lc.

$\frac{1}{r}(1,-2)$ $r_P = r$ $md_P X = -1 + \frac{1}{r}$

Q: Fix $(d,a) \in \mathbb{N} \times [-1,\infty)$

$\exists ?\ r(d,a)$ s.t. $\left[\begin{array}{l} P \in X \text{ dim } d,\ md_P X = a \\ \Rightarrow r_P \leq r(d,a) \end{array}\right.$ ($\to X$ lc.)

typical case: $a = -1$ (i.e. P a lc center) $r(2,-1) = 6$

(Ishii-Fujino): $r(3,-1) = 66$

$E \subset Y$, $\hat{E} \subset \hat{Y}$, $f: Y \to X$, $\hat{f}: \hat{Y} \to \hat{X}$, $\hat{X} \to X$ index 1 cone, $\hat{X}$ Gor

$K_{\hat{E}} = \underbrace{K_{\hat{Y}} + \hat{E}}_{= \hat{f}^* K_{\hat{X}}}|_{\hat{E}} \sim 0$ $\hat{E} \curvearrowleft \mathbb{Z}^r$

Modification to canonical sings:

Q': fix $(d,a) \in \mathbb{N} \times [0,\infty)$

$\exists ?\ r'(d,a)$ s.t. $\left[\begin{array}{l} P \in X \text{ cano. dim } d,\ md_P X = a \\ \Rightarrow r_P \leq r'(d,a) \end{array}\right.$

Typical case: $a = 0$, i.e. P is a crepant center. $r(2,0) = 1$

Thm: $P \in X$ cano. dim 3. P is a crepant center ($\overset{def}{\iff} md_P X = 0$) $\Rightarrow r_P \leq 6$.
(in part, one can take $r'(3,0) = 6$).

Cor: Q' yes for $d = 3$.

In dim 3, $md_P X = 0, \frac{1}{r}, 2$
$r'(3,0) = 6$, $r'(3,\frac{1}{r}) = r$!, $r'(3,2) = 1$

Rmk: (1) Shokurov conjectured: $r_P \le 6$ (presumably because for $P \in S \times \mathbb{A}^1 = X$, $r_P \le 6$, $\mathrm{md}_P X = 0$ not cano.
(lc.)

We have examples only for $r_P \le 4$:

(Morrison, Ishida-Iwashita): P cyc. quotient singularity $\Rightarrow r_P \le 3$.
$r_P \ne 1 \Leftarrow \underbrace{\frac{1}{4n}(1, 2n+1, -2), \frac{1}{14}(1,9,11)}_{r_P=2}, \underbrace{\frac{1}{9}(1,4,7)}_{r_P=3}$

(Hayakawa-Takeuchi): $P \underset{\text{isol}}{\in} (f=0) \subset \mathbb{C}^4/\mathbb{Z}_r \Rightarrow r_P \le 4$
$r_P = 4 \Leftrightarrow (x_1x_2 + x_3^3 + x_4^2 = 0) \subset \frac{1}{8}(1,5,3,7)$

(2): need: P crepant center.
e.g. $\overset{P\in X \simeq}{0} \in (x_1x_2 + x_3^2 = 0) \subset \frac{1}{r}(1,-1,0,1)$. cano, $\mathrm{md}_P X = \frac{1}{r}$, crepant center x_4-axis.

Sketch of pf. of theorem

Crepant partial resolution: $Y \xrightarrow{f} X$ (exists by min. model theory)
term. $K_Y = f^*K_X$
$f^{-1}P \subset$, $\ni P$

- Take $0 \ne E \subset f^{-1}P$ reduced divisor.

$\Rightarrow$ characterization of r_P as follows: $f_* \mathcal{O}_Y(\underbrace{iK_Y}_{= f^*K_X} - E) = \begin{cases} m_P \cdot \mathcal{O}_X(iK_X), & r_P \mid i \\ \mathcal{O}(iK_X), & r_P \nmid i. \end{cases}$ (projection formula)
$i \in \mathbb{Z}$

If $r_P \nmid i$, $f_*\mathcal{O}(iK_Y - E) = \{u \in K_X \mid (u)_Y + iK_Y - E \ge 0\}$ ($K_Y = f^*K_X$)
$\cap$
$\mathcal{O}(iK_X) \simeq \{u \in K_X \mid \underbrace{(u)_X + iK_X}_{\text{not Cartier at } P} \ge 0\}$ $\Rightarrow \exists D > 0$ st. $(u)_X + iK_X \ge D$ ($P \in$)

Hence we get "$\cup$" containment above as well. That is, $\boxed{\mathcal{O}(iK_X) = f_*\mathcal{O}(iK_Y - E)}$ $(*)$

- find $E > 0$ satisfying $(*)$ and vanishing $R^j f_*\mathcal{O}(iK_Y - E) = 0$ $\forall j \ge 1$ $\forall i$.

- $0 \to f_*\mathcal{O}(iK_Y - E) \to f_*\mathcal{O}(iK_Y) \to f_*\mathcal{O}(iK_Y|_E) \to 0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; Q_i$

Then $\dim Q_i = \chi(iK_Y) - \chi(iK_Y - E)$.
$\| \begin{cases} 1, & r_P \mid i \\ 0, & r_P \nmid i \end{cases}$, explicitly described by Reid's singular Riemann Roch
(usual term + contribution term from $Q = \frac{1}{r_Q}(1,-1,b_Q)$ from Y.

- $r_P = \operatorname{lcm}_Q \{r_Q\}$

- $\dim Q_0 - \dim Q_1 = \sum_Q \frac{v_Q(r_Q - v_Q)}{2r_Q} \rightsquigarrow r_P \le 9. \overset{\text{finer argument}}{\Rightarrow} r_P \le 6$, qed
$\|$
1 (unless $r_P = 1$)

Rmk: $Y \xrightarrow{f}$ (terminal) $P \in X$ lc. $K_Y + \Delta = f^*K_X$. Then sing RR $\rightsquigarrow$ $1 = \underbrace{\sum_Q \frac{v_Q(r_Q - v_Q)}{2r_Q}}_{\text{can not control!}} + (\text{neg term})$