Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Charles Favre

Talk Title: Uniform Izumi's theorem

Date: 05 / 09 / 2013 **Time:** 11 : 30 am / pm (circle one)

List 6-12 key words for the talk: ______________________________

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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Uniform Izumi's theorem

Charles Favre
Ecole Polytechnique

Joint work with S. Boucksom and M. Jonsson.

Izumi's theorem states that any two divisorial valuations centered at a smooth point on an affine variety are comparable up to a constant factor.

We study the variation of this constant when the valuations vary in a suitable way.

Uniform Izumi's Theorem

— joint w/ Boucksom & Jonsson

$X^n/\mathbb{C}$ affine, $0 \in X$ smooth, $v : \mathbb{C}[X] \to \mathbb{R}$ valuation, $v|\mathbb{C}^\times = 0$,
$v(m_{X,0}) := \min\{v(f) \mid f \in m_{X,0}\} > 0$ $\quad v = \mathrm{ord}_0$

Divisorial valuation: $X \xleftarrow{\pi} X_\pi$ bir. morphism, iso above $X \setminus \{0\}$, $E \subseteq \pi^{-1}(0)$ $\quad v := \mathrm{ord}_E$

Thm (Izumi): v div. valuation centered @ closed pt.) $\exists C > 1$, $\forall f \in \mathbb{C}[X]$, $\frac{1}{C}\mathrm{ord}_0(f) \le v(f) \le C \cdot \mathrm{ord}_0(f)$.

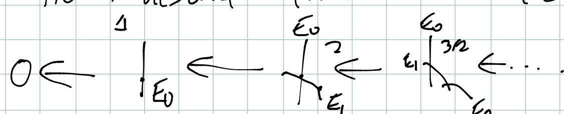
(lower bd is easy; $f \in m^{\mathrm{ord}_0 f}_{X,0}$, so $v(m_{X,0}) \cdot \mathrm{ord}_0(f) \le v(f)$).

- "analytic". Skoda, Tougeron
  $|f|^{-\frac{1}{\mathrm{ord}_0(f)+\varepsilon}} \in L^2$ for suff. small ε; C = log discrepancy of E.
  (algebraized by Ein-Lazarsfeld-Smith)
- "combinatorial". uses "key polynomials"
  F—Jonsson
  nd: ~~
- "geometric": Izumi, Rees, Hübl-Swanson, Boddani

Def: $\alpha(v) := \sup_{f \in m_{X,0}} \frac{v(f)}{\mathrm{ord}_0(f)} \in \mathbb{R}^\times_+$.

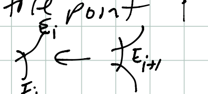
Question: Study the variation of $\alpha(v)$ when v varies.

ex = How to describe $\alpha(v)$ in the surface case?



... Then $v = \mathrm{ord}_{E_n}$, where E_n is the last one.

$\alpha_i := \alpha(\mathrm{ord}_{E_i})$, $\beta_i := \mathrm{ord}_{E_i}(m_{X,0})$.

- you blow up a free point  $\alpha_{i+1} = \alpha_i + \frac{1}{\beta_i}$
- satellite point  $\frac{\alpha_i + 1}{\beta_i + \beta_j} = \frac{\alpha_i}{\beta_i} + \frac{\alpha_j}{\beta_j}$

Topology on {divisorial valuations} $v_n \to v \iff v_n(f) \to v(f)\ \forall f$

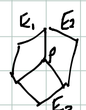
$p_n \ne p_m$, $p_n \in E_0$, $v_n = \mathrm{ord}_{p_n}$, $\underset{v_n \to \mathrm{ord}_0}{\alpha(v_n) = 2} \not\to \alpha(\mathrm{ord}_0) = 1$.

ex: Monomial valuations.

Fix $X=(x_1,\dots,x_n)$, $s=(s_1,\dots,s_n)\in \mathbb{R}_+^n$. $\quad v_{X,s}\left(\sum a_I x^I\right)=\min\{I\cdot s \mid a_I\neq 0\}$

Then $\alpha(v_{X,s})=\max\{s_i \mid 1\le i\le n\}$. This is a piecewise linear fn.

$0\in X \xleftarrow{\pi} X_\pi,\ \pi^{-1}(0)=\bigcup E_i$, $\pi^{-1}(0)$ SNC, $E_I:=\bigcap_{i\in I} E_i$ irreducible, $E_I\neq\emptyset \rightsquigarrow \sigma_I=\left\{\begin{matrix}\text{mon. vals.}\\ \text{w.r.t. } E_i,\, i\in I\end{matrix}\right\}$

$0\in \mathbb{A}^3 \xleftarrow{\pi}$ 

$I=\{1,2,3\}$, $p=E_{123}$

z_1,z_2,z_3 at p $\quad E_1=\{z_1=0\}$

$\sigma_{123}=\{\pi_* v_{z,s} \mid s\in \mathbb{R}_+^3\}$.

Thm: $\alpha:\sigma_I\simeq(\mathbb{R}_+)^{|I|}\longrightarrow \mathbb{R}_+$:
- 1-homogeneous ($\alpha(tv)=t\alpha(v)$)
- concave
- Lipschitz-continuous.

illustration: $I=\{1,2\}$, $\sigma_I=(\mathbb{R}_+^2)$,  $(\mathbb{R}_+)^2$

Volume of a valuation: $\mathfrak{a}(v,t)=\{v\geq t\}$ $\ t>0$ $\ \mathfrak{m}$-primary.

$\lim_{t\to\infty}\dfrac{e_{HS}(\mathfrak{a}(v,t))}{t^n}=:\mathrm{vol}(v)$

ELS, Mustaţă, Cutkosky: This notion coincides w/ the usual notion of volume.

Thm: $\mathrm{vol}^{-1}:\sigma_I\longrightarrow\mathbb{R}_+$ is n-homogeneous. (i.e. $\mathrm{vol}^{-1}(tv)=t^n\mathrm{vol}^{-1}(v)$)

$\mathrm{vol}^{-1}: \sigma_I\cap\{v(\mathfrak{m})=1\}\longrightarrow\mathbb{R}_+$ is Lipschitz continuous.

Thm 1 $\Rightarrow$ Thm 2:

$\mathrm{vol}^{-1}(v_{z,s})-\mathrm{vol}^{-1}(v_{z,s'})\ \leftarrow$

$1-A\,\mathrm{dist}(v_s,\mathrm{ad}_0)\le \dfrac{v_s}{v_{s'}}\le 1+A\,\mathrm{dist}(v_s,\mathrm{ad}_0)$

$1-A|s-s'|\le \dfrac{v_s}{v_{s'}}\le 1+A|s-s'|$

$v_s\le(1+A|s-s'|)v_{s'}$

$\mathrm{vol}^{-1}(v_s)\le(1+A|s-s'|)^n\mathrm{vol}^{-1}(v_{s'})$

Sketch of pf. of thm 1:

* geometric pf. of Izumi's theorem

$0 \xleftarrow{\pi} E_0,\ E_1$ WTS $\mathrm{ord}_{E_1}(f) \le C\cdot \mathrm{ord}_0(f)$, $f\in\mathbb{C}[x]$

$\pi^*[f=0] = \underset{\text{strict transform}}{Z(f)} + \mathrm{ord}_0(f)E_0 + \mathrm{ord}_{E_1}(f)E_1$. A ample divisor, $A^{n-2}\cdot E_0$.

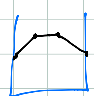
$0 = (\ge 0 \text{ term}) + \mathrm{ord}_0(f)\cdot(\overset{<0}{E_0|_{E_0}\cdot A^{n-2}}) + \mathrm{ord}_{E_1}(f)\,(\overset{>0}{E_0\cdot E_1\cdot A^{n-2}})$.

In the end, $\mathrm{ord}_{E_1}(f) \le \mathrm{ord}_0(f)\cdot \dfrac{-E_0|_{E_0}\cdot A^{n-2}}{E_0\cdot E_1\cdot A^{n-2}}$

* Lipschitz continuity:

$\alpha(v_s) = \sup_f \dfrac{v_s(f)}{\mathrm{ord}_0(f)}$.

Pick $f\in\mathbb{C}[x]$, $s\longmapsto \dfrac{v_s(f)}{\mathrm{ord}_0(f)} = \min\{I s,\ a_I\neq 0\}$ piecewise linear and affine



Goal = prove that the derivatives of $s\longmapsto \dfrac{v_s(f)}{\mathrm{ord}_0(f)}$ at the boundary pts are bounded.