MSRI
Mathematical Sciences Research Institute

17 Gauss Way   Berkeley, CA 94720-5070   p: 510.642.0143   f: 510.642.8609   www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Holger Brenner

Talk Title: Something is irrational in Hilbert-Kunz theory

Date: 05 / 10 / 2013 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: ______________________

Please summarize the lecture in 5 or fewer sentences: Monsky asked whether the Hilbert-Kunz multiplicity of a ring can ever be irrational. One may expand the question to Hilbert-Kunz multiplicities of $\mathfrak{m}$-primary ideals, or even more generally, to finite-length modules (defined by Seibert). Using techniques arising from the algebraic geometry of vector bundles, the speaker finds an example where the Hilbert-Kunz multiplicity of a finite-length module is irrational.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Something is irrational in Hilbert-Kunz theory

— in progress

Setting: R comm Noeth. ring containing a field of char $p>0$, $F: R \to R$ Frobenius homo $f \mapsto f^p$, $q=p^e$

$(F^e(I))\cdot R = I^{[q]} = (f_1^q, \ldots, f_n^q)$ whenever $I=(f_1, \ldots, f_n)$.

R is local or graded (w/ maximal ideal m)

If I is m-primary, R/I is Artinian (i.e. supported only at m) so R/I has finite length.

$R/I^{[q]}$ also has finite length. $HKF(I,e) := \ell(R/I^{[q]})$ (Kunz 1969)

Monsky 1983: $\lim_{e\to\infty} \dfrac{\ell(R/I^{[q]})}{q^{\dim R}} = e_{HK}(I) \in \mathbb{R}$, the Hilbert-Kunz multiplicity of R

$e_{HK}(R)=1 \iff R$ is regular (Watanabe-Yoshida)

Suspicion: $e_{HK}(R) \in \mathbb{Q}$.

Rationality results:

- If R is regular (or even if I has finite proj dim)
- Special equations $X_1^{d_1} + \cdots + X_n^{d_n}$ (Monsky, Han, Gessel) (m)
- Cayley cubic (Buchweitz-Chen)
- Semigroup rings (Conca, Watanabe) (m)
- invariant rings (Watanabe-Yoshida)
- dim 2 graded (B., Trivedi)

Conjecture (Monsky): $R := \mathbb{Z}/(2)[X,Y,Z,U,V]/(UV+X^3+Y^3+XYZ)$. Then $e_{HK}(m) = \frac{4}{3} + \frac{5}{14\sqrt{7}}$ irrational!

II. Generalized Hilbert-Kunz theory

$R \xrightarrow{F^e} R = {}^eR$. $\quad R/I \otimes_R {}^eR = R/I^{[q]} = F^{e*}(R/I)$. Instead of cyclic

modules, allow any module of finite length. That is,

Def: $e_{HK}(M) = \lim_{q\to\infty} \dfrac{\lambda(F^{e*}M)}{q^d}$, $d=\dim R$, for any M of finite length.

Question: $\overset{?}{\in} \mathbb{Q}$

What if M is not of finite length? (cf. Hilbert-Kunz criterion for tight closure).

$f \in I^* \iff e_{HK}(I) = e_{HK}(I+(f))$, I m-primary.

Epstein-Yao:

$$\lim_{q\to\infty} \frac{\lambda(H_m^0(F^{e*}M))}{q^{\dim R}} ?$$

Dao-Smirnov

$$\lim_{q\to\infty} \frac{\lambda(H^i_m(F^{e*}M))}{q^{\dim R}}$$

Thm (Dao-Smirnov): R Cohen-Macaulay on U = punctured spectrum, M locally free on U, then their limit exists for all $i<d$.

R standard-graded normal CM, isol. singularity, $K=\bar{K}$, $Y=\operatorname{Proj} R$, $\mathcal{O}_Y(1)$, $\dim Y = d$

Goal: Relate HK $\sim$ geometric data on Y

$$\begin{tikzcd}[column sep=small]
& 0 \arrow[d] & & 0 & 0 \arrow[d] & & 0 \\
& & Syz_2 \arrow[ur] \arrow[dr] & & & Syz_1 \arrow[ur] \arrow[dr] & \\
\cdots \arrow[r] & F_3 \arrow[rr] \arrow[ur] & & F_2 \arrow[rr] \arrow[ur] & & F_1 \arrow[r] & F_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

F_i free, complex exact on U, $F_i = \bigoplus R(-\beta_{ij})$

Thm: $d=\dim Y$

$$e_{HK}(M) = \lim_{q\to\infty} \frac{\sum_{m=0}^{\infty} h^d(F^{e*}Syz_d)(m)}{q^{d+1}} + \frac{H^d}{(d+1)!}\left(\sum_{i=0}^{d}(-1)^{d+1-i}\left(\sum_{j\in g_i} \beta_{ij}^{d+1}\right)\right),$$ where H^d = self-intersection # and

$h^d(S) = \dim_K H^d(Y,S)$ for a sheaf S.

Rmk: $d=1$, $\dfrac{h^1(F^{e*}S)}{q^2}$

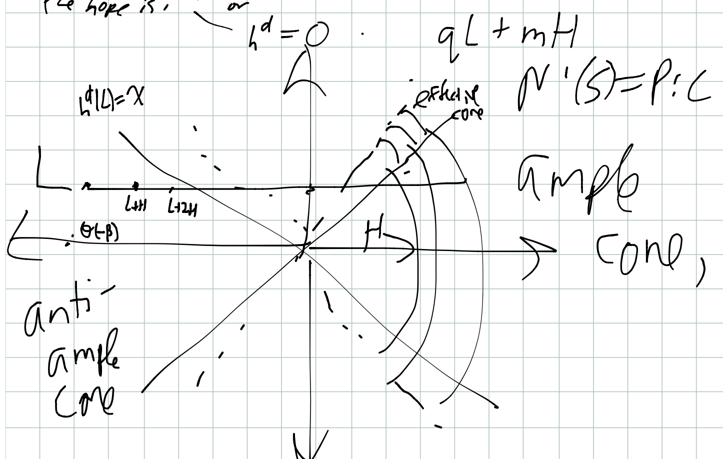
$Syz_d = \bigoplus \mathcal{O}(-\beta_{d+1,j})$

Let $F.$ be minimal res. Then Syz_d is a maximal Cohen-Macaulay module.

IV. "The splitting case"

If $Syz_d = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, $\mathcal{L}_i$ line bundles, $\lim_q \dfrac{\sum_{m=0}^{\infty} h^d(\mathcal{L}^q(m))}{q^d} = ?$

The hope is: $h^d = \chi$ or $h^d = 0$.



Def: $b(L) = \sup\{\frac{m}{n} \mid nL + mH \text{ antiample}\} =:$ "antiample threshold".

Thm: Suppose eff cone = $\overline{\text{ample cone}}$. L line bundle. $b = b(L)$

$$\lim_{q\to\infty} \frac{\sum_{m=0}^{\infty} h^d(\mathcal{L}^q(m))}{q^{d+1}} = \frac{b}{d!}\left(\sum_{i=0}^{d} \frac{1}{i+1}\binom{d}{i} b^i L^{d-i} H^i\right).$$

Cor: Assumptions as in the thm. Assume $b \notin \mathbb{Q}$. In the surface case,

$\lim_{e} \frac{\sum h^2(\mathcal{L}^q(m))}{q^3} = \frac{b}{2}\left(L^2 + b \cdot L \cdot H + \frac{1}{3} b^2 H^2\right)$ might be irrational.

[he reveals a board: Oguiso-Cayley surface. In these cases, $b \notin \mathbb{Q}$.]

VI: Rings

$R = K[x,y,z,w]/(F)$, $F =$ from the O-C surface, $\dim K >> 0$. There exists an R-module M such that $e^2_{HK}(M) := \lim_{e} \frac{\lambda(H^2_m(F^{e*}M))}{q^3}$ is irrational.

Cor: $\exists I \subseteq R$, not m-primary, s.t. $e^0_{HK}(I) = \lim_{q} \frac{\lambda(H^0_m(R/I^{[q]}))}{q^3}$ is irrational

Dao-Smirnov

Thm $\exists$ Artinian f.g. R-module M s.t. $e_{HK}(M) = \lim_{e \to \infty} \frac{\lambda(F^{e*}M)}{q^3}$ is irrational.

(one wants to show it for $M = R/I$ and even for $M = R/m$)