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**Name:** Neil Epstein **Email/Phone:** nepstei2@gmu.edu

**Speaker's Name:** Vikram Mehta

**Talk Title:** The singularities of the moduli spaces of vector bundles over curves in characteristic p

**Date:** 05 / 10 / 2013 **Time:** 2:00 am / pm (circle one)

**List 6-12 key words for the talk:** ____________

**Please summarize the lecture in 5 or fewer sentences:** (see abstract)

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# The Singularities of the Moduli Spaces of Vector Bundles over Curves in characteristic p (joint work with Vikraman Balaji and Venkata Balaji)

***Vikram B. Mehta***
*Tata Institute*

We discuss the singularities of these spaces via degeneration from characteristic zero and the notion of good filtrations for representations in characteristic p.

# The singularities of the moduli spaces of vector bundles over curves in characteristic $p$

— joint w/ Venkata Balaji ①, Vikraman Balaji ②, (③)

Let $X$ be a : compact Riemann surface / or: smooth proj. curve $/\mathbb{C}$ } of genus $g \geq 2$

$J$: line bundle of degree $0$
$J^d$: " " " " $d$
$J \to J^d$
$L \mapsto L \otimes M$

$\mathrm{Hom}(\pi_1^{top} X_{an}, S^1) = (S^1)$.

vector bundles of $rk > 1$: Mumford: $X$ smooth projective, dim $1$ / $k = \bar{k}$.

Def: $V$ is stable (resp. semistable) if $\forall\, 0 \subsetneq W \subsetneq V$, $\mu(W) := \frac{\deg W}{\operatorname{rank} W} < (\leq)\ \mu(V) = \frac{\deg V}{rk\, V}$.

N.S. over $\mathbb{C}$: $V$ vector bdl of degree $0$. Then: $V$ is stable (resp. semistable) $\Leftrightarrow$ $V \simeq V_\sigma$ for $\sigma \in \mathrm{Hom}(\pi_1^{top} X_{an}, U(V))$ and $\sigma$ is irreducible.

Def: $V$ is polystable if $V = \bigoplus$ stables of degree $0$
($\Leftrightarrow V \simeq V_\sigma$, $\sigma \in \mathrm{Hom}(\pi_1^{top}, U(V))$)

GIT construction of stables (resp semistables) & degree $0$. (let $N$ = its dimension)
Namely, $V$ s.s. rank $r$, deg $0$, $\exists m_0$ s.t. $\forall m \geq m_0$, $H^1(V(m)) = 0$, $H^0(V(m))$ generates $V(m)$

Hilbert scheme $H$: $\mathcal{O}_X^N \to F \to 0$, $GL(N)$ acts on $H$

Let $R^s$ ($R^{ss}$) be the open of $H$ s.t. $q \in R^s$ ($R^{ss}$)
$\Leftrightarrow$ 1) $\mathcal{O}_X^N \to F_q$, $F_q$ v.b. such that $(\ast\ast)$, and 2) $H^0(\mathcal{O}_X^N) \to H^0(F_q)$ is $\simeq$.

$\mathcal{U}(r)$ = good quotient of $R^{ss}$ mod $GL(N)$.

$R^{ss}$ $GL(N)$ inv. are $(A_i)$
$R^{ss} \to R^{ss}/\!/GL(N)$ problem is; orbits are not closed!

$\mathcal{U}(r)$ = normal proj variety
$\mathcal{U}(r)^s := R^s \to R^s/\!/GL(N)$
$\mathcal{U}^s := R^s \to R^s/\!/PGL(N)$ ; here the orbits are closed.

$R^s$ ($R^{ss}$) are nonsingular dim.
$0 \to K \to \mathcal{O}^N \to V \to 0$, $H^1(K^* \otimes V) = 0$, $H^0(K^* \otimes V) = r^2(g-1) + 1 + PGL(N)$

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In char $0$, $\mathcal{U}(r,0)$ has rat'l singularities (Hochster–Roberts, Boutot).

In char $p$, $\mathcal{U}^s$ is n.s.; what about $\mathcal{U}$?

$X_W \longrightarrow W$, $k, K$. Construct $\mathcal{U}_W, \mathcal{U}_W^s$ ?

$H_W$ = relative Hilbert scheme of quotients of $\mathcal{O}_{X_W}^N \to F \to 0$. $R_W^{ss} \supset R_W^s$. $F_k$ $F_K$

Construct $\mathcal{U}_W$: $R_W^{ss}/\!/GL(N,W)$, $\mathcal{U}_W^s := R_W^{ss}/\!/GL(N,W)$.

1) $\mathcal{U}_W \otimes_W k = \mathcal{U}_k$ ?  $\mathcal{U}_W^s \otimes_W k = \mathcal{U}_k^s$ ✓

2) $\mathcal{U}_W \otimes_W K = \mathcal{U}_K$ ? Yes.

Remarks: $\overset{\checkmark \text{ reduce mod } p}{\mathcal{U}_W/(p) = \mathcal{U}_k}$ ?

1) $\mathcal{U}_k \to \mathcal{U}_W/(p)$

2) $\mathcal{U}_W/(p)$ is integral.

3) $\mathcal{U}_k \longrightarrow \mathcal{U}_W/(p)$ is bijective on closed points.

We have to show that $\underline{\mathcal{U}_W/(p) \text{ is normal}}$

Suppose $V$ is a polystable bundle on $X_k$. ($V \cong A_1^{a_1} \oplus A_2^{a_2} \oplus \dots$, $A_i$ stable and pairwise non-isom). $\mathrm{Aut}(V) = \prod_i GL(a_i)$

$\mathrm{Aut}(V)$ acts on $H^1(X, \mathrm{End}\, V)$. good filtration: $G$ a reductive group/, $E$ fin. dim. $G$-module. Then $E$ has a good filtration if $\exists$ $0 = E_0 \subset E_1 \subset \dots \subset E_n = E$ s.t. $E_i/E_{i-1} \simeq H^0(G/B, L(\chi_i))$.

Thm (Andersen-Jantzen) $G_W$ reductive gp scheme$/W$, $E_W$ a $G_W$-module. Suppose that for each $n$, $S^n(E_k^\#)$ has a good filtration. Then $S^\cdot(E_W^\#)^{G_W}/(p) = S^\cdot(E_k^\#)^{G}_k$.

Let $V$ be polystable in char $p$. Any lift $V_W = \bigoplus A_{i,W}^{a_i}$ to char 0 is polystable. $H^1(\mathrm{End}\, V_W)/\!/\mathrm{Aut}\, V_W/(p)$

$= H^1(\mathrm{End}\, V_k)/\!/\mathrm{Aut}\, V_k$

$\Rightarrow \mathcal{U}_k \longrightarrow \mathcal{U}(W)/(p)$ is an iso. Now apply a thm of Hashimoto

Thm (Hashimoto): $G$ acts on v.s. $V$ (char $p$), $S^n(V^\#)$ is good th. Then $S^n(V^\#)^G$ is strongly F-regular ($\Rightarrow$ Cohen Macaulay)

$\mathcal{U}_k$ in char $p$ is also Gorenstein: $\mathcal{U}_k$, $T$ = worst pt. = trivial bundle ETS $T = \mathcal{O}_X^r$.
But $\mathcal{O}_{\mathcal{U}_k, T} = H^1(\mathrm{End}\, T)/\!/\mathrm{Aut}\, T = M(r)/GL(r,k) = M(r)^{\oplus g}/\!/PGL(r)$.

$\underline{M_G,\ M_G^g}$

$M_{SL(r)}$ $SU(r)$ $R_0^{ss}$ = fiber of map $(R^{ss} \longrightarrow J)$. $Y \subset X$ $Y/\!/G \subset X/\!/G$.

$0 \to \mathrm{End}^0(V) \longrightarrow \mathrm{End}(V) \to \mathcal{O}_X \to 0$

char 0: if $p \nmid r$, $H^0(-) \Rightarrow H^0 = 1 \to 1 \to 0$

$H^1 \to$

$(r^2-1)(g-1)$ $\quad r^2(g-1)+1$

char $p \mid r$ $\begin{cases} \mathrm{End}^0 \to \mathrm{End}\, V \to \mathcal{O}_X \\ H^0 \xrightarrow{\sim} H^0 \to H^0(\mathcal{O}_X) \\ H^1 \to H^1 \to H^1(X, \mathcal{O}_X) \end{cases}$ $H^1(X, \mathrm{End}^0 V) \ni 1$ $\quad T = \dfrac{H^1(X, \mathrm{End}^0 V)}{H^0(X, \mathcal{O}_X)} = \ker(H^1(\mathrm{End}\, V) \longrightarrow H^1(X, \mathcal{O}_X))$