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Name: Neil Epstein **Email/Phone:** nepstei2@gmu.edu

Speaker's Name: Nobuo Hara

Talk Title: Stabilization of the Frobenius push-forward and the F-blowup sequence

Date: 05 / 10 / 2013 **Time:** 3:30 am / pm (circle one)

List 6-12 key words for the talk: ______

Please summarize the lecture in 5 or fewer sentences: (see abstract)

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Stabilization of the Frobenius push-forward and the F-blowup sequence

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Given a variety X in characteristic $p>0$, the e-th F-blowup of X is defined to be the universal birational flattening of the e-times iterated Frobenius push-forward $F^e_*O_X$ of the structure sheaf. The stabilization of the list of indecomposable summands of $F^e_*O_X$ as e varies implies the stabilization of the of the F-blowup sequence. We will discuss these two properties in the simplest non-trivial cases, emphasizing computations in concrete examples.

Stabilization of the Frobenius pushforward and the F-blowup sequence

$X/k=\bar{k}$ char $p>0$, or a localization or completion of X
variety

$F: X \circlearrowleft$, $F^e_* \mathcal{O}_X$ = "Frobenius pushforward" $= \mathcal{O}_X^{1/p^e}$.

$\operatorname{rank}_{\mathcal{O}_X}(\mathcal{O}_X^{1/p^e}) = p^{e \cdot \dim X}$

Kunz: X nonsingular $\iff F^e_*(\mathcal{O}_X)$ flat $\forall e$ (equiv. for some $e>0$)

F-blowup (T. Yasuda): let $M \in Coh(X)$.
1) A modification $f: Y \to X$ is called a flattening of M if $f^\star M = f^*M/_{\text{torsion}}$ is flat.
2) A flattening f is universal if $\forall$ flattenings $g: Z \to X$ of M, g factors through f. That is:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, dashed, "\exists"'] & X \\ Y \arrow[ur, "f"'] \arrow[ur, phantom, shift left=3, "\circlearrowleft"] & \end{tikzcd} \qquad Y := Bl_M(X),$$

3) For $e \geq 0$, the e^{th} F-blowup := the universal flattening of $F^e_* \mathcal{O}_X$. (isom on X_{sm})
- written $FB_e(X) \xrightarrow{\varphi_e} X$.

Known Results: $X = \operatorname{Spec} R$

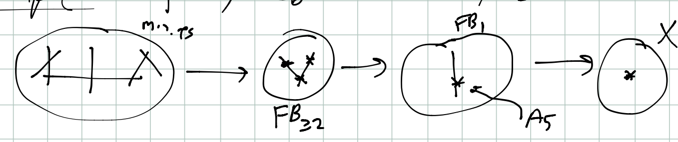
Thm (Yasuda, Toda-Yasuda): If $Y/G = X$ is a tame quotient singularity (i.e. $G \circlearrowright Y$, G finite group, $Y = \operatorname{Spec} S \xrightarrow{\pi} X = \operatorname{Spec} R$, $R = S^G$, S a RLR), then $FB_e(X) \cong Hilb^G(Y) \cong Bl_{\pi_* \mathcal{O}_Y}(X)$, $\forall e \gg 0$

Thm: R 2-dim'l F-regular ring. $FB_e(X) \cong$ the minimal resolution, $e \gg 0$.

In both cases, can construct a finite cover by a regular scheme.

Prop: let R be a 2-dim'l rational singularity. Then $FB_e(X)$ is normal and dominated by $M.R.$ minimal resolution

Example: $p=2$, $E_6^0: z^2 + x^3 + y^2 z = 0$



Observation: F-pure and FFRT $\Rightarrow$ stabilization of FB-sequences.
(F-pure $\iff F^e_* \mathcal{O}_X \overset{\oplus}{\hookrightarrow} F^{e+1}_* \mathcal{O}_X$, so $\exists$ seq. of morphisms $\cdots \to FB_{e+1}(X) \to FB_e(X) \to \cdots \to X$ "monotone" (Yasuda))

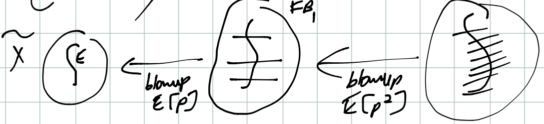
Assume R is complete local.
FFRT means $\{$indec R-modules that occur as $\oplus$'and of R^{1/p^e}, $\exists e = 0,1,2,\ldots\}/\cong$ is finite
$= \{M_1, \ldots, M_n\}$.

It follows that for $e \gg 0$, $FB_e X \cong Bl_M X$, $M = M_1 \oplus \cdots \oplus M_n$.

Example: let (X, x) be a simple elliptic singularity. This is not FFRT.
min. res. $\uparrow$
$(\tilde{X}, E)$, $E = \pi^{-1}(x)$ elliptic curve.

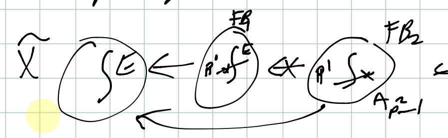
Thm: (H.-Sawada-Yasuda, H): The following are equivalent (under the above conditions):
(1) The F-blowup sequence stabilizes
(2) $FB_e X \cong \tilde{X}$ $\forall e \geq 1$
(3) $-E^2 \neq p^0, p^1, p^2, \ldots$

Example:
E ordinary (X is F-pure)



$\leftarrow \cdots\cdots$, $-E^2 = p = 3$.

E supersingular



$\leftarrow \cdots$ not monotone.

Problem: Does the FB sequence of F-regular singularities stabilize?

A. Singh: $\exists$ non-FFRT F-regular singularity in dimension 7.

Global: X smooth proj. var/k. X is globally FFRT (GFFRT) $\overset{def}{\iff}$ $\{$indec summands of $F^e_* \mathcal{O}_X\}_{/\cong}$ is finite. ($\exists e \geq 0$)

Example:
1) dim 1 case: $F^e_* \mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(-1)^{\oplus p^e - 1}$, hence $\mathbb{P}^1$ GFFRT.

But: $g(X) \geq 1 \Rightarrow X$ not FFRT. $F^e_* \mathcal{O}_X = \bigoplus_{P \in E[p^e]} \mathcal{O}_X(P - P_0)$

X s.s. g $\geq 2 \Rightarrow F^e_* \mathcal{O}_X$ indec.

2) X toric $\Rightarrow F^e_* \mathcal{O}_X = \bigoplus (\text{l.b.'s})$ FFRT (Achinger: $F^e_* L = \bigoplus(\text{l.b.})$)

$\circledast$ $\pi: X \to \mathbb{P}^2$ blowup at $P_1, \ldots, P_n \in \mathbb{P}^2$

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^2 \\ \cup & & \cup \\ E_i & \longrightarrow & P_i \end{array}$$

Experimental result: ($p = 2, n = 4$). Then X is GFFRT.

- $F^e_* \mathcal{O}_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}(-1)^{\oplus \cdots} \oplus \mathcal{O}(-2)^{\oplus \cdots}$
- $F^e_* \mathcal{O}_{\tilde{U}} \cong \mathcal{O}_{U^2}^{\oplus q(q+1)/2} \oplus \mathcal{O}_{\tilde{U}}(E)^{\oplus q(q-1)/2}$

$\cup$

$\pi^* F^e_* \mathcal{O}_U = \mathcal{O}_{\tilde{U}}^{\oplus q^2}$

$$\begin{array}{ccccc} -1 & -1 & -2 & -2 & -2 \\ -1 & -1 & -1 & -2 & -2 \\ -1 & -1 & -1 & -1 & -2 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \end{array}$$

$$0 \to \pi^* F^e_* \mathcal{O}_{\mathbb{P}^2} \to F^e_* \mathcal{O}_X \to \bigoplus_{i=1}^{n} \mathcal{O}_{E_i}(-1)^{\oplus \frac{q(q-1)}{2}} \to 0$$

$\circledast$

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{N} \to 0$$

$\mathcal{E}$: $\oplus$ of $\mathcal{O}^{r_0}, \mathcal{O}(-1)^{r_1}, \mathcal{O}(-2)^{r_2}$; $\mathcal{N}$: $\oplus$ of $\mathcal{O}_{E_i}(-1)^{\oplus s_i}$'s

$$\mathcal{E} \in \operatorname{Ext}^1(\mathcal{W}, \pi^*\mathcal{E}) \cong H^0(\mathcal{E}xt^1(\quad)) = k^{rs}$$

$$\bigoplus \text{ of } \underbrace{\mathcal{E}xt^1(\mathcal{O}_{E_i}(-1), \mathcal{O}_{\mathbb{P}^1}(-1))}_{\cong\, \mathcal{O}_{E_i}}$$

$$\begin{array}{c c}
\left[\begin{array}{cccc} \, & \, & \, & \, \\ \, & \, & \, & \, \\ \, & \, & \, & \, \end{array}\right] & \begin{array}{c} r_0 \\ r_1 \\ r_2 \end{array} \\
\begin{array}{cccc} s_1 & s_2 & \cdots & s_n \end{array} &
\end{array}$$

Lem: 1)

Example: $p=2, n=4$ $\quad \mathcal{O}_X, \ \mathcal{O}_X(E_i) \otimes \pi^*\mathcal{O}(-1), \ i=1,\dots,4.$

$$\mathcal{E}(\mathcal{G}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} r_1 = 1 \\ r_2 = 1 \end{array}$$

$$\mathcal{E}(\mathcal{B}) = \left.\begin{bmatrix} 1 & & & 1 \\ & 1 & & 1 \\ & & 1 & 1 \end{bmatrix}\right\} r_1 = 3 \quad (r_0 = r_2 = 0)$$

$$\mathcal{E}(\mathcal{F}) = \left[\begin{array}{ccc|ccc|ccc|ccc} & & & & & & & & & 1 & 1 & 1 \\ \hline 1 & & & 1 & & & 1 & & & 1 & & \\ & 1 & & & 1 & & & 1 & & & 1 & \\ & & 1 & & & 1 & & & 1 & & & 1 \end{array}\right] \begin{array}{l} r_1 = 1 \\ r_2 = 3 \end{array}$$

$E_* \mathcal{G} \cong \mathcal{G}^*$

so X is GFFRT.